

Block Orthogonal Polynomials: II. Hermite and Laguerre Standard Block Orthogonal Polynomials

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Abstract

The standard block orthogonal (SBO) polynomials $P_{i;n}(x)$, $0 \leq i \leq n$ are real polynomials of degree n which are orthogonal with respect to a first Euclidean scalar product to polynomials of degree less than i . In addition, they are mutually orthogonal with respect to a second Euclidean scalar product. Applying the general results obtained in a previous paper, we determine and investigate these polynomials when the first scalar product corresponds to Hermite (resp. Laguerre) polynomials. These new sets of polynomials, we call Hermite (resp. Laguerre) SBO polynomials, provide a basis of functional spaces well-suited for some applications requiring to take into account special linear constraints which can be recast into an Euclidean orthogonality relation.

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1. Introduction

In a first paper [1] (paper I), within the general frame of an Euclidean vector space endowed with distinct scalar products, and using the Gram-Schmidt orthogonalization (G-SO) process, we investigate the general properties of *standard block orthogonal* (SBO) *polynomials*. By definition, for a given non-negative integer i , these real polynomials $P_{i;n}(x)$, $n = i, i+1, \dots$, of degree n , are

(i) orthogonal to the i -dimensional subspace \mathcal{P}_i of real polynomials of degree less than i with respect to a first Euclidean scalar product (\cdot, \cdot) defined by a non-negative weight function $w(x)$ on the interval $[a, b]$,

$$(x^m, P_{i;n}) := \int_a^b dx w(x) x^m P_{i;n}(x) = 0 \quad m = 0, \dots, i-1 \quad n = i, i+1, \dots; \quad (1.1)$$

(ii) mutually orthogonal with respect to a second Euclidean scalar product $(\cdot, \cdot)_2$, similarly defined by a distinct non-negative weight function $w_2(x)$ on the same interval ¹,

$$(P_{i;m}, P_{i;n})_2 := \int_a^b dx w_2(x) P_{i;m}(x) P_{i;n}(x) = H_{i;n} \delta_{m,n} \quad m, n = i, i+1, \dots. \quad (1.2)$$

In other words, for a given subspace \mathcal{P}_i , there exists a unique *orthogonal complement* \mathcal{P}_i^\perp of codimension i , orthogonal with respect to the scalar product (\cdot, \cdot) , and complementary with respect to the space of real polynomials \mathcal{P} of any degree ²,

$$(\mathcal{P}_i, \mathcal{P}_i^\perp) = 0 \quad \text{and} \quad \mathcal{P} = \mathcal{P}_i \oplus \mathcal{P}_i^\perp. \quad (1.3)$$

Then, $\{P_{i;i}, P_{i;i+1}, \dots\}$ is an orthogonal basis of \mathcal{P}_i^\perp with respect to the scalar product $(\cdot, \cdot)_2$. These basis polynomials are defined uniquely apart from a constant multiplicative factor for each polynomial.

¹ More generally, the scalar product (\cdot, \cdot) can be defined by the positive measure $\{\mathcal{D}, d\mu\}$ where, μ is a non-constant and non-decreasing real function on the real domain \mathcal{D} . Similarly, $(\cdot, \cdot)_2$ can be defined by $\{\mathcal{D}_2, d\mu_2\}$. See section 3.2 in paper I.

² See equation (3.4) in paper I, written with \mathcal{P}_N instead of \mathcal{P} and valid for any non-negative integer value of N .

Classical orthogonal polynomials (i.e., the Hermite, Laguerre and Jacobi polynomials) are orthogonal with respect to a scalar product defined with a particular positive measure. These polynomials arise frequently and have been studied in great detail [2–5]³. The purpose of this paper is to determine and investigate the SBO polynomials in the special cases where the first measure, $\{[a, b], w(x)dx\}$ corresponds to the Hermite and then the Laguerre polynomials. This work has been motivated by the recent consideration of ‘*constrained orthogonal polynomials*’ by Giraud *et al* [6–8]. Their physical motivations, sketched in section 1 of paper I, were basically to define orthogonal basis functions, taking into account special linear constraints (i.e. orthogonality relations), to provide a functional space well-suited for some specific applications. Thus, aiming to use the Hohenberg-Kohn variational principal [9] for the ground state energy in nuclear physics, the variable function, i.e. the particle density, has to satisfy the constraint of particle number conservation. In toy models, expanding the particle density on SBO polynomials, these authors only consider the case $i = 1$ (and partly $i = 2$) for the Hermite polynomials in [6, 8] and the Laguerre polynomials in [7]. Our works generalize these results to all non-negative integer values of i and provide a systematic study of the properties of these polynomials, in particular getting new recurrence and differentiation formulae.

This paper is organized as follows. Explicit expressions and properties of the Hermite (resp. Laguerre) SBO polynomials are given in section 2 (resp. 3), i.e. for: the connection coefficients in section 2.2 (resp. 3.2), the differentiation formulae in section 2.3 (resp. 3.3), the recurrence relations with respect to the degree n in section 2.4 (resp. 3.4) and with respect to both i and n in section 2.5 (resp. 3.5), the special value at $x = 0$ in section 2.6 (resp. 3.6), the properties of the zeros in section 2.7 (resp. 3.7), and finally, the interrelation between the Hermite and the Laguerre cases in section 3.8. We give our conclusions in section 4. For completeness and to illustrate how the general methods used allow to recover well-known results in a unified way, several properties of the Hermite (resp. Laguerre) polynomials are briefly recalled in appendix A (resp. B). This aims also at emphasizing the similarities to and the differences from the study of the classical orthogonal polynomials and the SBO polynomials. Some expressions related to the Pochhammer symbol and generalized hypergeometric series are computed in appendix C. Some determinants with entries expressed in terms of gamma function, except possibly for the last row, are evaluated in appendix D. Finally, some explicit special cases for the Hermite (resp. Laguerre) SBO polynomials are listed in appendix E (resp. F).

Throughout the remaining of this paper, the same conventions and notations as in paper I are used⁴, in addition

- references to (sub)sections, appendices and equations of paper I are preceded by I-;
- $[x]$ denotes the largest integer less than or equal to the real number x ;
- for any function $f : x \mapsto f(x)$, the first (resp. second) derivative with respect to x is denoted by a prime, f' (resp. a double prime, f'').

2. Hermite standard block orthogonal polynomials

Definition and main properties of Hermite polynomials, H_n , $n = 0, 1, \dots$, are recalled in appendix A. They correspond to the measure defined by

$$a := -\infty \quad b := \infty \quad w := e^{-x^2} . \quad (2.1)$$

2.1. Metric tensor components for a particular weight function w_2

(i) *Basic quantity* $\gamma_{j,k}$. With the same interval, $[a, b]$, let us choose,

$$w_2 = w_2^{(\mu)} := e^{-\mu x^2} \quad \mu > 0 . \quad (2.2)$$

³ The standard textbook is [2]. See, also [3] chapter 22, [4] sections 10.6–10.13 or [5] section 8.9.

⁴ See, [1] at the end of section 1. In particular, it is recalled that: - *monic polynomials*, i.e. with the coefficient of the highest power equal to one, and also any related quantities are denoted by hatted letters, e.g., \hat{P}_n and \hat{h}_n ; - classical orthogonal polynomials are defined according to [2] chapter 22, [3] chapter X or [4] section 8.9.

This choice is motivated by the application considered in [6–8] for the special value $\mu = 2$. Our study will be restricted in (ii) below to this special case, since it allows to push the calculation to its end analytically.

As shown in section I-3.3.3, w and w_2 being even with $a = -b$, the SBO polynomial $P_{i;n}$, $n = i, i+1, \dots$, as the Hermite polynomials, are even or odd according as n is even or odd, equation I-(3.41),

$$P_{i;n}(-x) = (-1)^n P_{i;n}(x). \quad (2.3)$$

Actually, the even and odd polynomials separate from each other and one has, equation I-(3.50),

$$\text{if } (-1)^{i+n} = 1 \quad \text{then} \quad \hat{P}_{i-1;n} = \hat{P}_{i;n} \quad 0 \leq i-1 \leq n. \quad (2.4)$$

From equation I-(3.12), the basic quantity $\gamma_{j,k}(\mu)$ reads

$$\gamma_{j,k}(\mu) := (H_j, H_k)_2 = \int_{-\infty}^{\infty} dx e^{-\mu x^2} H_j(x) H_k(x) \quad (2.5)$$

which vanishes by parity if $j+k$ is odd. Expanding the Hermite polynomials in terms of powers of x with the coefficients given by equation (A.7) and using equation (A.4), one finds

$$\begin{aligned} \gamma_{2j,2k}(\mu) &= (-1)^{j+k} 2^{2(j+k)} \Gamma\left(\frac{1}{2} + j\right) \Gamma\left(\frac{1}{2} + k\right) \\ &\quad \times \sum_{p=0}^j \sum_{q=0}^k \binom{j}{p} \binom{k}{q} \frac{(-1)^{p+q}}{\Gamma\left(\frac{1}{2} + p\right) \Gamma\left(\frac{1}{2} + q\right)} \int_{-\infty}^{\infty} dx e^{-\mu x^2} x^{2(p+q)} \\ &= (-1)^{j+k} 2^{2(j+k)} \Gamma\left(\frac{1}{2} + j\right) \Gamma\left(\frac{1}{2} + k\right) \mu^{-\frac{1}{2}} \\ &\quad \times \sum_{p=0}^j \sum_{q=0}^k \binom{j}{p} \binom{k}{q} \frac{\Gamma\left(\frac{1}{2} + p + q\right)}{\Gamma\left(\frac{1}{2} + p\right) \Gamma\left(\frac{1}{2} + q\right)} (-\mu)^{-(p+q)}. \end{aligned} \quad (2.6)$$

The value of the sum over p and q above is considered in lemma C.2 (appendix C). It can be expressed in terms of an *hypergeometric function* or the function S defined by equation (C.8), then,

$$\gamma_{2j,2k}(\mu) = (-1)^{j+k} 2^{2(j+k)} \Gamma\left(\frac{1}{2} + j\right) \Gamma\left(\frac{1}{2} + k\right) \mu^{-\frac{1}{2}} S\left(j, k, \frac{1}{2}; -\mu^{-1}, -\mu^{-1}\right). \quad (2.7)$$

For the odd polynomials, from equation (A.8), a similar calculation yields

$$\gamma_{2j+1,2k+1}(\mu) = (-1)^{j+k} 2^{2(j+k+1)} \Gamma\left(\frac{3}{2} + j\right) \Gamma\left(\frac{3}{2} + k\right) \mu^{-\frac{3}{2}} S\left(j, k, \frac{3}{2}; -\mu^{-1}, -\mu^{-1}\right). \quad (2.8)$$

Alternately, these last two equations can be obtained using, with some care, the definite integrals given in [5, 10]⁵. Another way to get the basic quantity $\gamma_{j,k}(\mu)$ is to compute the scalar product $(F_s, F_t)_2$ where, $F_z(x)$ is the generating function of Hermite polynomials given by equation (A.29). Then, $\gamma_{j,k}(\mu)$ is related to the coefficient of $s^j t^k$ in the expansion of this scalar product, i.e.,

$$\begin{aligned} (F_s, F_t)_2 &= \int_{-\infty}^{\infty} dx e^{-\mu x^2} F_s(x) F_t(x) = \sum_{j,k=0}^{\infty} \frac{1}{j! k!} \gamma_{j,k}(\mu) s^j t^k \\ &= \left(\frac{\pi}{\mu}\right)^{\frac{1}{2}} \exp \frac{2st + (1-\mu)(s^2 + t^2)}{\mu}. \end{aligned} \quad (2.9)$$

⁵ See, [5] equation 7.374 5 where, the second argument of the hypergeometric function should be $-n$ instead of n (the result has to be symmetric in m and n) and in addition for α^2 real, α has to be taken positive. See also [10] 2.20.16 1 where, the factor $(\pm 1)^{m+n}$ should be replaced by $(-1)^{m+n}$, or else, 2.20.16 6. In both cases, one has to use also the linear transformation formula [3] 15.3.6.

For $\mu = 1$, i.e. for $w_2 = w$, one has to recover the orthogonality condition of Hermite polynomials. Indeed, from equation (C.14),

$$S(j, k, \frac{1}{2}; -1, -1) = \frac{j!}{\Gamma(\frac{1}{2} + j)} \delta_{j,k} \quad (2.10)$$

whence, with the *duplication formula* [3] ⁶,

$$\gamma_{2j,2k}(1) = 2^{4j} j! \Gamma(\frac{1}{2} + j) \delta_{j,k} = 2^{2j} (2j)! \pi^{\frac{1}{2}} \delta_{j,k} \quad (2.11)$$

recovering thereby the value of h_{2j} given by equation (A.6). The calculation is similar for $\gamma_{2j+1,2k+1}(1)$. The results follows readily also from equation (2.9), indeed, $(F_s, F_t)_2 = \pi^{\frac{1}{2}} e^{2st} = \sum_{j=0}^{\infty} h_j(st)^j / (j!)^2$.

(ii) *The special case* $w_2 = w_2^{(2)} = e^{-2x^2}$. When $\mu = 2$, one gets simple expressions. Indeed, from equation (C.15) setting $z = -\frac{1}{2}$, one has

$$\gamma_{2j,2k}(2) = (-1)^{j+k} 2^{j+k-\frac{1}{2}} \Gamma(\frac{1}{2} + j + k) \quad (2.12)$$

$$\gamma_{2j+1,2k+1}(2) = (-1)^{j+k} 2^{j+k+\frac{1}{2}} \Gamma(\frac{3}{2} + j + k) \quad (2.13)$$

or, equivalently [5] ⁷,

$$\gamma_{j,k}(2) = \begin{cases} (-1)^{[j/2]+[k/2]} 2^{\frac{1}{2}(j+k-1)} \Gamma(\frac{1}{2}(j+k+1)) & \text{if } j+k \text{ even} \\ 0 & \text{otherwise.} \end{cases} \quad (2.14)$$

Alternately, this result follows also from equation (2.9),

$$(F_s, F_t)_2 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(s-t)^2} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{m=0}^{2n} (-1)^{n-m} 2^{-n} \frac{(2n)!}{n! m! (2n-m)!} s^m t^{2n-m}. \quad (2.15)$$

Setting $j = m$, $k = 2n - m$ and using the duplication formula to get $(2n)!/n! = \pi^{-\frac{1}{2}} 2^{2n} \Gamma(\frac{1}{2} + n)$, one recovers equation (2.14).

From now on, except explicitly noted, only this special case $\mu = 2$ will be considered.

Then, the corresponding polynomials $P_{i;n}$, $0 \leq i \leq n$ are called the *Hermite SBO polynomials*. The case $i = 1$ has already been considered in [6, 8] with different choices of weight functions and normalization. (See footnote 50 in appendix E for interrelations with our polynomial $\hat{P}_{1;n}$.) The case $i = 2$, only for the odd polynomials, has been also considered in [6], as already discussed in the conclusion of paper I (see footnote 44 of paper I). (See footnote 51 for interrelations with our polynomial $\hat{P}_{2;2n+1}$.)

2.2. Connection coefficients between block orthogonal and classical polynomials

The relevant quantities are given by equations I-(3.42)–I-(3.47) in terms of determinants with the (j, k) -element of the type $\Gamma(c + j + k)$, except possibly for the last row. This kind of determinant is evaluated in lemma D (appendix D). From equations I-(3.43), I-(3.46), (2.12), (2.13) and (D.3), one finds for $0 \leq i \leq n$,

$$Z_{i;n}^{(e)} = 2^{(i+n-\frac{1}{2})(n-i+1)} \prod_{j=i}^n (j-i)! \Gamma(\frac{1}{2} + i + j) \quad Z_{i;i-1}^{(e)} := 1 \quad (2.16)$$

$$Z_{i;n}^{(o)} = 2^{(i+n+\frac{1}{2})(n-i+1)} \prod_{j=i}^n (j-i)! \Gamma(\frac{3}{2} + i + j) \quad Z_{i;i-1}^{(o)} := 1 \quad (2.17)$$

⁶ See, e.g., [3] 6.1.18, $\Gamma(z)\Gamma(\frac{1}{2} + z) = 2^{-2z+1} \pi^{\frac{1}{2}} \Gamma(2z)$.

⁷ Equation (2.12) is given in [5] equation 7.374 2, with the wrong sign: the result should be symmetric in j and k and positive for $j = k$.

and therefore, from equations I-(3.44) and I-(3.47) with k_n given by equation (A.3) [6]⁸,

$$\hat{H}_{2i-1;2n} = \hat{H}_{2i;2n} = 2^{-(2n+\frac{1}{2})} (n-i)! \Gamma\left(\frac{1}{2} + i + n\right) \quad (2.18)$$

$$\hat{H}_{2i;2n+1} = \hat{H}_{2i+1;2n+1} = 2^{-(2n+\frac{3}{2})} (n-i)! \Gamma\left(\frac{3}{2} + i + n\right). \quad (2.19)$$

The monic SBO polynomials follow from equations I-(3.42), I-(3.45) and (D.2), for $0 \leq i \leq n$,

$$\hat{P}_{2i-1;2n} = \hat{P}_{2i;2n} = \sum_{m=i}^n 2^{-(n+m)} \binom{n-i}{m-i} \frac{\Gamma(\frac{1}{2} + i + n)}{\Gamma(\frac{1}{2} + i + m)} H_{2m} \quad (2.20)$$

$$\hat{P}_{2i;2n+1} = \hat{P}_{2i+1;2n+1} = \sum_{m=i}^n 2^{-(1+n+m)} \binom{n-i}{m-i} \frac{\Gamma(\frac{3}{2} + i + n)}{\Gamma(\frac{3}{2} + i + m)} H_{2m+1} \quad (2.21)$$

where, according to equations I-(3.13), (2.4) and (A.3), for $i \geq 0$,

$$\hat{P}_{i;i} = \hat{H}_i := 2^{-i} H_i \quad \hat{P}_{i;i+1} = \hat{H}_{i+1} = 2^{-(i+1)} H_{i+1}. \quad (2.22)$$

The orthogonality and normalization of these polynomials with respect to the scalar product $(\cdot, \cdot)_2$ can be checked directly using equation (C.14), e.g., for the even polynomials with equations (2.20) and (2.14),

$$\begin{aligned} & \int_{-\infty}^{\infty} dx e^{-2x^2} \hat{P}_{2i;2n}(x) \hat{P}_{2i';2n'}(x) \\ &= 2^{-(n+n'+\frac{1}{2})} \Gamma\left(\frac{1}{2} + i + n\right) \Gamma\left(\frac{1}{2} + i' + n'\right) S(n-i, n'-i, \frac{1}{2} + 2i; -1; -1) \\ &= 2^{-(2n+\frac{1}{2})} (n-i)! \Gamma\left(\frac{1}{2} + i + n\right) \delta_{n,n'} \end{aligned} \quad (2.23)$$

recovering equation (2.18).

Equations (2.20) and (2.21) can be inverted to give for $0 \leq i \leq n$,

$$H_{2n} = \sum_{m=i}^n (-1)^{n-m} 2^{n+m} \binom{n-i}{m-i} \frac{\Gamma(\frac{1}{2} + i + n)}{\Gamma(\frac{1}{2} + i + m)} \hat{P}_{2i;2m} \quad (2.24)$$

$$H_{2n+1} = \sum_{m=i}^n (-1)^{n-m} 2^{1+n+m} \binom{n-i}{m-i} \frac{\Gamma(\frac{3}{2} + i + n)}{\Gamma(\frac{3}{2} + i + m)} \hat{P}_{2i+1;2m+1}. \quad (2.25)$$

As for the Hermite polynomials in appendix A, these relations can be obtained either using the general formulae I-(A.22) and I-(A.28) where, the determinants which occur can be evaluated with equation (D.2), or directly, using lemma A (appendix A), e.g., for the even polynomials, setting $j = m - i$, $k = n - i$, $\rho_j = 2^{-(i+j)} \Gamma(\frac{1}{2} + 2i + j)$ and $\sigma_j = (-1)^j 2^{i+j} \Gamma(\frac{1}{2} + 2i + j)$.

It should be noted that the connection coefficients relating the polynomials $\hat{P}_{i;n}$ and H_n in equations (2.20), (2.21), (2.24) and (2.25) have the same structure as the connection coefficients relating H_n and the monomials in equations (A.7), (A.8), (A.12) and (A.13), respectively. This is due to the fact that in both cases, the metric tensor components are of the same type $\Gamma(c + j + k)$, apart from some factors depending separately on the indices j and k , see equations (2.14) and (A.4). (See also equation (2.30) below.)

As already noted in section 3.2 of paper I (see equations I-(3.18) and I-(3.19)), for $i = 0$, $\hat{P}_{0;n}$ has to be the unique standard monic orthogonal polynomial $\hat{Q}_{2;n}$ corresponding to the weight function w_2 . Since $w_2^{(2)}(x) = w(\sqrt{2}x)$, with the normalization (A.3) and changing x into $\sqrt{2}x$ in the integral defining the scalar product of Hermite polynomials, one gets for $n \geq 0$,

$$\hat{P}_{0;n}(x) = 2^{-\frac{1}{2}n} \hat{H}_n(\sqrt{2}x) = 2^{-\frac{3}{2}n} H_n(\sqrt{2}x). \quad (2.26)$$

⁸ These equations generalize the formulae given in [6] (4) (resp. (19)) for the special case $i = 1$ (resp. $i = 2$ for the odd polynomials) with different choices of w , w_2 and normalization, see equation (E.12) and footnote 50 (resp. 51)

Thereby, setting $i = 0$ in equations (2.20), (2.21), (2.24) and (2.25), using the duplication formula and changing m into $n - m$ in the sum over m yield the connection coefficients between the two basic sets of polynomials $\{H_m, m = 0, \dots, n\}$ and $\{H_m(\sqrt{2}x), m = 0, \dots, n\}$ for $n \geq 0$,

$$H_n(\sqrt{2}x) = \sum_{m=0}^{[n/2]} 2^{\frac{1}{2}(n-2m)} \frac{n!}{m!(n-2m)!} H_{n-2m}(x) \quad (2.27)$$

$$H_n(x) = \sum_{m=0}^{[n/2]} (-1)^m 2^{-\frac{1}{2}n} \frac{n!}{m!(n-2m)!} H_{n-2m}(\sqrt{2}x). \quad (2.28)$$

These remarkable relations can be checked using the integral [5, 10]⁹ and the orthogonality relation of Hermite polynomials with h_n given by equation (A.6), e.g., one has

$$\int_{-\infty}^{\infty} dx e^{-x^2} H_n(\sqrt{2}x) H_{n-2m}(x) = 2^{\frac{3}{2}(n-2m)} \frac{n!}{m!} \pi^{\frac{1}{2}} = 2^{\frac{1}{2}(n-2m)} \frac{n!}{m!(n-2m)!} h_{n-2m}. \quad (2.29)$$

Another way to get the relation (2.27) is as follows. From equations (A.4) and (2.14), one has

$$(x^j, x^k) = (-1)^{[j/2]+[k/2]} 2^{-\frac{1}{2}(j+k-1)} \gamma_{j,k}(2) = \sqrt{2} \left((-1)^{[j/2]} 2^{-\frac{1}{2}j} H_j, (-1)^{[k/2]} 2^{-\frac{1}{2}k} H_k \right)_2 \quad (2.30)$$

namely, the metric tensor components of the scalar product $(,)$ in the basis $\{x^n, n = 0, 1, \dots\}$ and the metric tensor components of the scalar product $(,)_2$ in the basis $\{(-1)^{[n/2]} 2^{-\frac{1}{2}n} H_n, n = 0, 1, \dots\}$ are equal apart from a constant factor. Hence, while G-SO with respect to $(,)$ of the basis $\{x^n, n = 0, 1, \dots\}$ defines the Hermite polynomial H_n , G-SO with respect to $(,)_2$ of the basis $\{(-1)^{[n/2]} 2^{-\frac{1}{2}n} H_n, n = 0, 1, \dots\}$ defines the polynomial $\hat{P}_{0,n}$. In other words, the linear mapping defined by $x^n \mapsto (-1)^{[n/2]} 2^{-\frac{1}{2}n} H_n, n = 0, 1, \dots$, is such that $H_n \mapsto \propto \hat{P}_{0,n}, n = 0, 1, \dots$. Therefore, apart from a multiplicative factor c , $\hat{P}_{0,n}$ reads as a linear combination of $(-1)^{[m/2]} 2^{-\frac{1}{2}m} H_m, m = 0, 1, \dots$ with *the same connection coefficients* as in the expansion of H_n in terms of the monomials¹⁰, i.e. from equation (A.9),

$$\hat{P}_{0;n} = c \sum_{m=0}^{[n/2]} \frac{(-1)^m 2^{n-2m}}{m!(n-2m)!} \left((-1)^{[(n-2m)/2]} 2^{-\frac{1}{2}(n-2m)} H_{n-2m} \right). \quad (2.31)$$

With $\hat{P}_{0,n}$ a monic polynomial, computing the coefficient of x^n fixes $c = (-1)^{[n/2]} 2^{-\frac{3}{2}n} n!$, recovering thereby equation (2.27) with (2.26).

From equations (2.4) and (2.26), one has

$$\hat{P}_{1;2n+1}(x) = \hat{P}_{0;2n+1}(x) = 2^{-3(n+\frac{1}{2})} H_{2n+1}(\sqrt{2}x) \quad (2.32)$$

(see also the comments about equation I-(3.52) and in the conclusion of paper I, $\hat{P}_{1;2n+1} = \hat{Q}_{2;2n+1}$). This relation can be checked setting $i = 0$ in equation (2.21) and comparing with the expansion of $H_{2n+1}(\sqrt{2}x)$ given by equation (2.26).

It follows from equations I-(3.10), I-(3.11), (2.20), (2.21) and (A.10) that,

$$\hat{R}_{i;n} = 0 \quad (2.33)$$

$$\hat{S}_{2i-1;2n} = \hat{S}_{2i;2n} = -\frac{1}{4} \left(i(2i-1) + n(2n-1) \right) \quad (2.34)$$

$$\hat{S}_{2i;2n+1} = \hat{S}_{2i+1;2n+1} = -\frac{1}{4} \left(i(2i+1) + n(2n+1) \right). \quad (2.35)$$

⁹ See [5] equation 7.374 4 or [10] 2.20.16.7. p 503.

¹⁰ It is underlined in section I-2.1 that all the relevant quantities in G-SO procedure depend only on the metric tensor components.

For $i = 0$, one gets from above $\widehat{S}_{0;n} = -n(n-1)/8$, which can be checked using equations (2.26) and (A.10).

For the general properties of SBO polynomials about the *projection operators*, the *integral representations* and the *normalizations*, see sections 3.3.1, 3.3.2 and 3.3.4 in paper I, respectively.

2.3. Differentiation formulae

Let us summarize well-known results for all classical orthogonal polynomials:

- as for any standard orthogonal polynomials, the *three term recurrence formula* I-(3.55) holds;
- the weight function which defines the scalar product satisfies a differential equation. This implies several differentiation formulae, e.g., the *linear homogeneous differential equation of the second order*¹¹,

$$A(x)y'' + B(x)y' + \lambda_n y = 0 \quad (2.36)$$

where, $A(x)$ and $B(x)$ are independent of n and λ_n is independent of x .

All these relations follow from the three term recurrence formula and one of the first order differentiation formulae. Moreover, among the three formulae—the recurrence equation, a first order differentiation equation and the second order differential equation—any pair of equations implies the third one. In particular, two first order differentiation formulae or, the first and second order differentiation formulae, imply the recurrence formula. This is the method used to get a five term recurrence formula in the next subsection. However, while the differentiation formulae for classical orthogonal polynomials can be derived in two ways, as recalled in appendices A (item 4) and B (item 4), the second method only can be used for block orthogonal polynomials. This is due to the fact, already encountered in sections 3.3.5 and 3.3.7 of paper I, that the scalar product $(\cdot, \cdot)_0$, defined by equation I-(3.5)¹², is no longer defined by a single integral. Here, for a given nonzero i , one has to deal with both scalar products (\cdot, \cdot) and $(\cdot, \cdot)_2$, each of them defined by an integral with a different weight function.

For the scalar product (\cdot, \cdot) , the weight function w satisfies the differential equation (A.17), which implies the relations (A.18) and (A.19). Similarly, for the scalar product $(\cdot, \cdot)_2$ associated with the weight function $w_2^{(\mu)}$ defined by equation (2.2), one has for any polynomials f and g ,

$$w_2^{(\mu)'} = -2\mu x w_2^{(\mu)} \quad (2.37)$$

$$(f, g')_2 = -(f' - 2\mu x f, g)_2 \quad (2.38)$$

$$(f'' - 2\mu x f', g)_2 = -(f', g')_2 = (f, g'' - 2\mu x g')_2. \quad (2.39)$$

(i) For $0 \leq \ell < i$, the polynomial H'_ℓ of degree $\ell - 1$ is in the subspace \mathcal{P}_i . Therefore, from the orthogonality condition (1.1), H'_ℓ and $\widehat{P}_{i;n}$ for $i \leq n$ are orthogonal with respect to the scalar product (\cdot, \cdot) , and from equation (A.18), one has¹³,

$$0 = (\widehat{P}_{i;n}, H'_\ell) = -(\widehat{P}'_{i;n} - 2x\widehat{P}_{i;n}, H_\ell). \quad (2.40)$$

Hence, the particular polynomial $\widehat{P}'_{i;n} - 2x\widehat{P}_{i;n}$ of degree $n + 1$ is orthogonal to the subspace \mathcal{P}_i with respect to the scalar product (\cdot, \cdot) , i.e. it belongs to the orthogonal subspace \mathcal{P}_i^\perp , see equation (1.3), and thus reads $\sum_{j=i}^{n+1} \alpha_{i;n,j} \widehat{P}_{i;j}$. Computing the coefficient of x^{n+1} in $\widehat{P}'_{i;n} - 2x\widehat{P}_{i;n}$ fixes $\alpha_{i;n,n+1} = -2$. By parity, $\alpha_{i;n,n}, \alpha_{i;n,n-2}, \dots$ vanish. Now, with respect to the scalar product $(\cdot, \cdot)_2$, one has from equations (1.2) and (2.38), for $j = i, \dots, n + 1$,

$$\begin{aligned} \alpha_{i;n,j} \widehat{H}_{i;j} &= (\widehat{P}_{i;j}, \widehat{P}'_{i;n} - 2x\widehat{P}_{i;n})_2 = (\widehat{P}_{i;j}, \widehat{P}'_{i;n})_2 - (2x\widehat{P}_{i;j}, \widehat{P}_{i;n})_2 \\ &= -(\widehat{P}'_{i;j} - 2(\mu - 1)x\widehat{P}_{i;j}, \widehat{P}_{i;n})_2. \end{aligned} \quad (2.41)$$

¹¹ See, e.g., [4] 10.6 (ii) p 164.

¹² Any Euclidean scalar product $(\cdot, \cdot)_1$ defined on \mathcal{P}_i , together with $(\cdot, \cdot)_2$ defined on \mathcal{P}_i^\perp induce a new Euclidean structure on \mathcal{P} with the scalar product defined by $(p, q)_0 := (p_i, q_i)_1 + (p_i^\perp, q_i^\perp)_2$ for any $p = p_i + p_i^\perp \in \mathcal{P}$, $p_i \in \mathcal{P}_i$, $p_i^\perp \in \mathcal{P}_i^\perp$ and $q = q_i + q_i^\perp \in \mathcal{P}$, $q_i \in \mathcal{P}_i$, $q_i^\perp \in \mathcal{P}_i^\perp$.

¹³ Actually, equation (2.40) still holds for $\ell = i$. Since from equation (2.22), $\widehat{P}_{i;i} = \widehat{H}_i$, the component $\alpha_{i;n,i}$ considered in equation (2.41) below, vanishes. This result is obtained also by parity.

In the special case $\mu = 2$, it turns out that $\hat{P}'_{i;j} - 2(\mu - 1)x\hat{P}_{i;j}$ is the particular polynomial $\hat{P}'_{i;j} - 2x\hat{P}_{i;j}$, introduced above for $j = n$ (i.e. of degree $j + 1$ in the subspace \mathcal{P}_i^\perp), and which reads $\sum_{k=i}^{j+1} \alpha_{i;j,k} \hat{P}_{i;k}$. Thus, for $\mu = 2$, *considered only henceforth*, one has for $i \leq j \leq n + 1$,

$$\alpha_{i;n,j} \hat{H}_{i;j} = -\alpha_{i;j,n} \hat{H}_{i;n} \sum_{k=i}^{j+1} \delta_{k,n}. \quad (2.42)$$

For $n = i$, all the α -coefficients are known: $\alpha_{i;i,i} = 0$, $\alpha_{i;i,i+1} = -2$. For $n \geq i + 1$, it follows from above that for $n \geq i + 2$ and $j = i, \dots, n - 2$, $\alpha_{i;n,j}$ vanishes. It remains to consider the case $j = n - 1$. With $\alpha_{i;n-1,n} = -2$, equations (2.18) and (2.19), one gets (by inspection of the four cases according to the parity of i and n),

$$\kappa_{i;n} := \alpha_{i;n,n-1} = 2 \frac{\hat{H}_{i;n}}{\hat{H}_{i;n-1}} = \frac{1}{2} (n - (-1)^{i+n} i). \quad (2.43)$$

Finally, for $\mu = 2$ and $0 \leq i \leq n$, one finds the *first order differentiation formula*¹⁴,

$$\hat{P}'_{i;n} = -2\hat{P}_{i;n+1} + 2x\hat{P}_{i;n} + \kappa_{i;n}\hat{P}_{i;n-1} \quad (2.44)$$

which still holds for $n = i$, since then $\kappa_{i;n}$ vanishes. This equation has to be compared with equation (A.26). For $n = i$, both equations coincide, using equation (2.22). For $i = 0$, using equation (2.26), one gets

$$H'_n = -\frac{1}{2}H_{n+1} + xH_n + nH_{n-1} \quad (2.45)$$

which follows from, e.g., the differentiation formula (A.21) and the recurrence formula (A.16).

With the boundary value of $\hat{P}_{i;i}$, given in terms of Hermite polynomials by equation (2.22), the differentiation formula (2.44) can be used to get step by step $\hat{P}_{i;n}$ for all i and n with $0 \leq i \leq n$. One can also take advantage of the parity relation (2.4) to divide by two the number of polynomials to be computed. A table of polynomials $\hat{P}_{i;n}$ for $i = 0, 1, 2$ and $n = i, i + 1, \dots, i + 4$ is given in appendix E.

Following the same steps as above, for $0 \leq \ell < i \leq n$, the polynomial xH'_ℓ of degree ℓ is orthogonal to $\hat{P}_{i;n}$ with respect to the scalar product (\cdot, \cdot) , and from equation (A.18), one has

$$0 = (\hat{P}_{i;n}, xH'_\ell) = (x\hat{P}_{i;n}, H'_\ell) = -(x\hat{P}'_{i;n} + \hat{P}_{i;n} - 2x^2\hat{P}_{i;n}, H_\ell) = -(x\hat{P}'_{i;n} - 2x^2\hat{P}_{i;n}, H_\ell). \quad (2.46)$$

Hence, the particular polynomial $x\hat{P}'_{i;n} - 2x^2\hat{P}_{i;n}$ of degree $n + 2$ belongs to \mathcal{P}_i^\perp and reads $\sum_{j=i}^{n+2} \beta_{i;n,j} \hat{P}_{i;j}$. Computing the coefficient of x^{n+2} in $x\hat{P}'_{i;n} - 2x^2\hat{P}_{i;n}$ fixes $\beta_{i;n,n+2} = -2$. By parity, $\beta_{i;n,n+1}, \beta_{i;n,n-1}, \dots$ vanish. Now, with respect to the scalar product $(\cdot, \cdot)_2$, one has from equations (1.2) and (2.38), for $i \leq j \leq n + 1$,

$$\begin{aligned} \beta_{i;n,j} \hat{H}_{i;j} &= (\hat{P}_{i;j}, x\hat{P}'_{i;n} - 2x^2\hat{P}_{i;n})_2 = (x\hat{P}_{i;j}, \hat{P}'_{i;n})_2 - (2x^2\hat{P}_{i;j}, \hat{P}_{i;n})_2 \\ &= -(x\hat{P}'_{i;j} - 2(\mu - 1)x^2\hat{P}_{i;j}, \hat{P}_{i;n})_2 - \hat{H}_{i;n}\delta_{j,n}. \end{aligned} \quad (2.47)$$

In the special case $\mu = 2$, it turns out that $x\hat{P}'_{i;j} - 2(\mu - 1)x^2\hat{P}_{i;j}$ is the particular polynomial $x\hat{P}'_{i;j} - 2x^2\hat{P}_{i;j}$, introduced above for $j = n$ (i.e. of degree $j + 2$ in the subspace \mathcal{P}_i^\perp), and which reads $\sum_{k=i}^{j+2} \beta_{i;j,k} \hat{P}_{i;k}$. Thus, one has for $i \leq j \leq n + 1$,

$$\beta_{i;n,j} \hat{H}_{i;j} = \left(-\beta_{i;j,n} \sum_{k=i}^{j+2} \delta_{k,n} - \delta_{j,n} \right) \hat{H}_{i;n} \quad (2.48)$$

¹⁴ This generalizes the formula given in [6] (7) for the special case $i = 1$ with different choices of w and w_2 , see footnote 50.

and therefore, $\beta_{i;n,n} = -\frac{1}{2}$. Then, for $n = i$ and $n = i + 1$, all the β -coefficients are known: $\beta_{i;i,i} = -\frac{1}{2}$, $\beta_{i;i,i+1} = 0$, $\beta_{i;i,i+2} = -2$ and $\beta_{i;i+1,i} = 0$, $\beta_{i;i+1,i+1} = -\frac{1}{2}$, $\beta_{i;i+1,i+2} = 0$, $\beta_{i;i+1,i+3} = -2$. For $n \geq i + 2$, it follows from equation (2.48) that for $n \geq i + 3$ and $j = i, \dots, n - 3$, $\beta_{i;n,j}$ vanishes. It remains to consider the case $j = n - 2$. From $\beta_{i;n-2,n} = -2$ and equation (2.43), one gets

$$\beta_{i;n,n-2} = 2 \frac{\hat{H}_{i;n}}{\hat{H}_{i;n-2}} = \frac{1}{2} \kappa_{i;n} \kappa_{i;n-1}. \quad (2.49)$$

Finally, in the case $\mu = 2$, for $0 \leq i \leq n$ and $\kappa_{i;n}$ given by equation (2.43), one finds another *first order differentiation formula*,

$$x\hat{P}'_{i;n} = -2\hat{P}_{i;n+2} + \frac{1}{2}(4x^2 - 1)\hat{P}_{i;n} + \frac{1}{2}\kappa_{i;n}\kappa_{i;n-1}\hat{P}_{i;n-2} \quad (2.50)$$

which still holds for $n = i$ and $n = i + 1$, since then $\kappa_{i;n}\kappa_{i;n-1}$ vanishes. This equation, which relates polynomials having the same parity, has to be compared with equation (A.27). For $i = 0$, using equation (2.26), one gets

$$4xH'_n = -H_{n+2} + 2(2x^2 - 1)H_n + 4n(n-1)H_{n-2} \quad (2.51)$$

which follows, e.g., from the differentiation formula (A.27) and the recurrence formula (A.28).

Starting with the values of $\hat{P}_{i;i}$ and $\hat{P}_{i;i+1}$ given by equation (2.22), the differentiation formula (2.50) allows, as equation (2.44), to compute step by step $\hat{P}_{i;n}$ for all i and n with $0 \leq i \leq n$ (see appendix E).

(ii) Following the same arguments again, for $0 \leq \ell < i \leq n$, the polynomial $H''_\ell - 2xH'_\ell$ of degree ℓ is orthogonal to $\hat{P}_{i;n}$ with respect to the scalar product (\cdot, \cdot) , and from equation (A.19), one has

$$0 = (\hat{P}_{i;n}, H''_\ell - 2xH'_\ell) = (\hat{P}''_{i;n} - 2x\hat{P}'_{i;n}, H_\ell). \quad (2.52)$$

Hence, the particular polynomial $\hat{P}''_{i;n} - 2x\hat{P}'_{i;n}$ of degree n belongs to \mathcal{P}_i^\perp and reads $\sum_{j=i}^n \gamma_{i;n,j} \hat{P}_{i;j}$. Computing the coefficient of x^n in $\hat{P}''_{i;n} - 2x\hat{P}'_{i;n}$ fixes $\gamma_{i;n,n} = -2n$. By parity, $\gamma_{i;n,n-1}, \gamma_{i;n,n-3}, \dots$ vanish. With respect to the scalar product $(\cdot, \cdot)_2$, one has from equations (1.2), (2.38) and (2.39), for $i \leq j \leq n$,

$$\begin{aligned} \gamma_{i;n,j} \hat{H}_{i;j} &= (\hat{P}_{i;j}, \hat{P}''_{i;n} - 2x\hat{P}'_{i;n})_2 = (\hat{P}_{i;j}, \hat{P}''_{i;n} - 2\mu x\hat{P}'_{i;n})_2 + 2(\mu - 1)(x\hat{P}_{i;j}, \hat{P}'_{i;n})_2 \\ &= \left((\hat{P}''_{i;j} - 2(\mu - 1)x\hat{P}'_{i;j}) - 2\mu(x\hat{P}'_{i;j} - 2(\mu - 1)x^2\hat{P}_{i;j}), \hat{P}_{i;n} \right)_2 - 2(\mu - 1)\hat{H}_{i;n}\delta_{j,n}. \end{aligned} \quad (2.53)$$

In the special case $\mu = 2$, once again, $\hat{P}''_{i;j} - 2(\mu - 1)x\hat{P}'_{i;j}$ (resp. $x\hat{P}'_{i;j} - 2(\mu - 1)x^2\hat{P}_{i;j}$) is the particular polynomial $\hat{P}''_{i;j} - 2x\hat{P}'_{i;j}$ (resp. $x\hat{P}'_{i;j} - 2x^2\hat{P}_{i;j}$), introduced above for $j = n$ (i.e. of degree j (resp. $j + 2$) in the subspace \mathcal{P}_i^\perp), and which reads $\sum_{k=i}^j \gamma_{i;j,k} \hat{P}_{i;k}$ (resp. $\sum_{k=i}^{j+2} \beta_{i;j,k} \hat{P}_{i;k}$). Thus, one has for $i \leq j \leq n$,

$$\gamma_{i;n,j} \hat{H}_{i;j} = \left(\gamma_{i;j,n} \sum_{k=i}^j \delta_{k,n} - 4\beta_{i;j,n} \sum_{k=i}^{j+2} \delta_{k,n} - 2\delta_{j,n} \right) \hat{H}_{i;n}. \quad (2.54)$$

For $n = i$ and $n = i + 1$, all the γ -coefficients are known: $\gamma_{i;i,i} = -2i$ and $\gamma_{i;i+1,i} = 0$, $\gamma_{i;i+1,i+1} = -2(i + 1)$. For $n \geq i + 2$, it follows from above that for $n \geq i + 3$ and $j = i, \dots, n - 3$, $\gamma_{i;n,j}$ vanishes. It remains to consider the case $j = n - 2$. From $\beta_{i;n-2,n} = -2$ and $\beta_{i;n,n} = -\frac{1}{2}$, one gets

$$\gamma_{i;n,n-2} = -4\beta_{i;n-2,n} \frac{\hat{H}_{i;n}}{\hat{H}_{i;n-2}} = 2\kappa_{i;n}\kappa_{i;n-1}. \quad (2.55)$$

Finally, in the case $\mu = 2$, for $0 \leq i \leq n$ and $\kappa_{i;n}$ given by equation (2.43), one finds the *second order differentiation formula* ¹⁵,

$$\hat{P}''_{i;n} - 2x\hat{P}'_{i;n} + 2n\hat{P}_{i;n} = 2\kappa_{i;n}\kappa_{i;n-1}\hat{P}_{i;n-2} \quad (2.56)$$

¹⁵ This generalizes the formula given in [6] (15) for the special case $i = 1$ with different choices of w and w_2 , see footnote 50.

which still holds for $n = i$ and $n = i + 1$, since then $\kappa_{i;n}\kappa_{i;n-1}$ vanishes. This equation has to be compared with the differential equation (A.24). For $n = i$ and $n = i + 1$, both equations coincide, using equation (2.22). For $i = 0$, from equation (2.26), one gets

$$H_n'' - x H_n' + n H_n = 2n(n-1) H_{n-2} \quad (2.57)$$

which follows, e.g., from equations (A.22) and (A.24).

2.4. Recurrence relation with respect to the degree n

In the case $\mu = 2$, eliminating $\hat{P}_{i;n}'$ between the two first order differentiation formulae (2.44) and (2.50) yields for $0 \leq i \leq n$ the *five term linear recurrence formula*,

$$\hat{P}_{i;n+2} - x \hat{P}_{i;n+1} + \frac{1}{4} \hat{P}_{i;n} + \frac{1}{2} \kappa_{i;n} x \hat{P}_{i;n-1} - \frac{1}{4} \kappa_{i;n} \kappa_{i;n-1} \hat{P}_{i;n-2} = 0 \quad (2.58)$$

where, $\kappa_{i;n}$ is defined by equation (2.43). This relation still holds for $n = i$ and $n = i + 1$, since $\kappa_{i;n}$ vanishes for $n = i$. It has to be compared with the three term recurrence formula (A.16). For $i = 0$, with equation (2.26), one finds

$$H_{n+2} - 2x H_{n+1} + 2 H_n + 4n x H_{n-1} - 4n(n-1) H_{n-2} = 0 \quad (2.59)$$

which follows from the three term recurrence formula (A.16).

Note that the five term recurrence formula (2.58) can as well be obtained from the first and second order differentiation formulae (2.44) and (2.56) as follows. Take the derivative of equation (2.44), eliminate $\hat{P}_{i;n}''$ using equation (2.56), and then eliminate the remaining first derivatives $\hat{P}_{i;n+1}'$, $\hat{P}_{i;n}'$ and $\hat{P}_{i;n-1}'$ using equation (2.44) again.

As the differentiation formulae (2.44) and (2.50), the recurrence formula (2.58) can be also used to get step by step $\hat{P}_{i;n}$ for all i and n with $0 \leq i \leq n$, starting from $\hat{P}_{i;i}$ (see appendix E).

2.5. Recurrence relations with respect to both i and n

(i) As shown in section 3.3.6 of paper I, SBO polynomials corresponding to different values of i can be related together. Thus, $\hat{P}_{2i+2;2n}$ can be expressed in terms of the polynomials $\hat{P}_{2i,2\ell}$ with $\ell = i, \dots, n$. From equations I-(3.58), (2.20) and (2.24), one has for $0 \leq i < n$,

$$\hat{P}_{2i+2;2n} = \sum_{\ell=i}^n \hat{P}_{2i,2\ell} \frac{2^{\ell-n}}{n-i} \binom{n-i}{n-\ell} \frac{\Gamma(\frac{3}{2} + i + n)}{\Gamma(\frac{1}{2} + i + \ell)} \sum_{m=\sup(i+1,\ell)}^n (-1)^{m-\ell} \binom{n-\ell}{m-\ell} \frac{m-i}{\frac{1}{2} + i + m} \quad (2.60)$$

where, the sum over m can be evaluated as follows. First, the lowest value of m can be taken as ℓ instead of $\sup(i+1, \ell)$, since the term added thereby occurs for $\ell = i$ only, and it vanishes for $m = i$. Then, with $p = n - \ell$, $q = m - \ell$, $\rho = \ell - i$ and $\sigma = \frac{1}{2} + i + \ell > 0$, the sum over m reads

$$\sum_{q=0}^p (-1)^q \binom{p}{q} \frac{q + \rho}{q + \sigma} = \delta_{p,0} + (\rho - \sigma) \sum_{q=0}^p (-1)^q \binom{p}{q} \frac{1}{q + \sigma} \quad (2.61)$$

where, using the binomial formula and the *beta function* [3]¹⁶,

$$\int_0^1 dx x^{\sigma-1} (1-x)^p = \sum_{q=0}^p (-1)^q \binom{p}{q} \frac{1}{q + \sigma} = B(\sigma, p+1) = \frac{\Gamma(\sigma)\Gamma(p+1)}{\Gamma(\sigma+1+p)}. \quad (2.62)$$

Putting all together, one finds for $0 \leq i < n$,

$$\hat{P}_{2(i+1);2n} = \hat{P}_{2i;2n} - 2^{-n} \left(\frac{1}{2} + 2i\right)(n-i-1)! \sum_{\ell=i}^{n-1} \frac{2^\ell}{(\ell-i)!} \hat{P}_{2i;2\ell}. \quad (2.63)$$

¹⁶ See, e.g., [3] 6.2.1 and 6.2.2.

Similarly for the odd polynomials, from equations (2.21) and (2.25), one has for $0 \leq i < n$,

$$\widehat{P}_{2(i+1)+1;2n+1} = \widehat{P}_{2i+1;2n+1} - 2^{-n} \left(\frac{3}{2} + 2i\right)(n-i-1)! \sum_{\ell=i}^{n-1} \frac{2^\ell}{(\ell-i)!} \widehat{P}_{2i+1;2\ell+1}. \quad (2.64)$$

With always the parity relation (2.4), these recurrence relations provide another way of computing $\widehat{P}_{i;n}$ for all $0 \leq i \leq n$, starting with the boundary values $\widehat{P}_{0;n}$ given by equation (2.26) (see appendix E). Thus, for $i = 0$, one gets for $n \geq 1$ [6]¹⁷,

$$\widehat{P}_{1;2n}(x) = \widehat{P}_{2;2n}(x) = 2^{-3n} H_{2n}(\sqrt{2}x) - 2^{-(n+1)}(n-1)! \sum_{\ell=0}^{n-1} \frac{2^{-2\ell}}{\ell!} H_{2\ell}(\sqrt{2}x) \quad (2.65)$$

$$\widehat{P}_{2;2n+1}(x) = \widehat{P}_{3;2n+1}(x) = 2^{-(3n+\frac{3}{2})} H_{2n+1}(\sqrt{2}x) - 2^{-(n+1)} 3(n-1)! \sum_{\ell=0}^{n-1} \frac{2^{-(2\ell+\frac{3}{2})}}{\ell!} H_{2\ell+1}(\sqrt{2}x) \quad (2.66)$$

i.e. expansions in terms of Hermite polynomials of the variable $\sqrt{2}x$, instead of x as given by equations (2.20) and (2.21).

(ii) On the right-hand side of equation (2.63), the sum over ℓ depends on n through the upper bound $n-1$ of the sum only. Therefore, replacing n by $n-1$ in equation (2.63) and eliminating $\sum_{\ell=i}^{n-2} \dots$ between the equation obtained thereby and equation (2.63) yield for $0 \leq i \leq n-1$,

$$4\widehat{P}_{2(i+1);2n} - 4\widehat{P}_{2i;2n} - 2(n-i-1)\widehat{P}_{2(i+1);2(n-1)} + (2i+2n-1)\widehat{P}_{2i;2(n-1)} = 0. \quad (2.67)$$

This relation still holds for $n = i+1$ since then, it reduces to equation (2.63). From equation (2.64), one gets a similar equation for the odd polynomials. Actually, both equations can be written as a remarkable single *linear four term recurrence relation, with coefficients independent of x , and which holds for all i and n having the same parity and such that $0 \leq i \leq n-2$,*

$$\text{if } (-1)^{i+n} = 1 \quad \text{then} \quad 4\widehat{P}_{i+2;n} - 4\widehat{P}_{i;n} - (n-i-2)\widehat{P}_{i+2;n-2} + (i+n-1)\widehat{P}_{i;n-2} = 0. \quad (2.68)$$

For $n = i+2$, this relation reduces to a three term recurrence relation. From equations (2.4) and (2.22), one finds for $i \geq 0$, the following expressions which can also be obtained from equations (2.20) and (2.21),

$$\widehat{P}_{i;i+2} = 2^{-(i+2)} (H_{i+2} + (1+2i)H_i) \quad (2.69)$$

$$\widehat{P}_{i+1;i+3} = \widehat{P}_{i;i+3} = 2^{-(i+3)} (H_{i+3} + (3+2i)H_{i+1}). \quad (2.70)$$

More generally, the four term recurrence relation (2.68) with the parity relation (2.4) provide another easy way of computing step by step $\widehat{P}_{i;n}$ for all i and n with $0 \leq i \leq n$, starting from the boundary values of $\widehat{P}_{i;i}$ given by equation (2.22) (see appendix E).

2.6. Special value at $x = 0$

The odd polynomials $\widehat{P}_{2i;2n+1} = \widehat{P}_{2i+1;2n+1}$ vanish at $x = 0$. For the even polynomials, from equations (2.20), (A.11) and (C.1), one finds for $0 \leq i \leq n$,

$$\begin{aligned} \widehat{P}_{2i-1;2n}(0) &= \widehat{P}_{2i;2n}(0) = 2^{-n} \sum_{m=i}^n (-1)^m 2^m \binom{n-i}{m-i} \frac{\Gamma(\frac{1}{2} + i + n)}{\Gamma(\frac{1}{2} + i + m)} \left(\frac{1}{2}\right)_m \\ &= (-1)^i 2^{-(n-i)} \left(\frac{1}{2}\right)_i \left(\frac{1}{2} + 2i\right)_{n-i} F\left(-(n-i), \frac{1}{2}; \frac{1}{2} + 2i; 2\right) \end{aligned} \quad (2.71)$$

¹⁷ Equation (2.65) has been already given in [6] (3) with different choices of w and w_2 , see footnote 50.

where, F is a terminating hypergeometric function, for which we don't know a closed form in terms of say gamma function products for all i and n . These values can be computed by recurrence as follows.

(i) *Boundary values for $n = i$ and $i = 0$.* From equations (2.22), (2.26) and (A.11), one gets

$$\widehat{P}_{2i;2i}(0) = 2^{-2i} H_{2i}(0) = (-1)^i \left(\frac{1}{2}\right)_i = (-1)^i \frac{(2i)!}{2^{2i} i!} \quad i = 0, 1, \dots \quad (2.72)$$

$$\widehat{P}_{0;2n}(0) = 2^{-3n} H_{2n}(0) = (-1)^n 2^{-n} \left(\frac{1}{2}\right)_n = (-1)^n \frac{(2n)!}{2^{3n} n!} \quad n = 0, 1, \dots \quad (2.73)$$

Setting for $0 \leq i \leq n$,

$$\widehat{P}_{2i;2n}(0) := (-1)^n 2^{-2(n-i)} \left(\frac{1}{2}\right)_i p_{i;n} = (-1)^n \frac{(2i)!}{2^{2n} i!} p_{i;n} \quad (2.74)$$

one finds the boundary values,

$$p_{i;i} = 1 \quad i = 0, 1, \dots \quad p_{0;n} = 2^n \left(\frac{1}{2}\right)_n = \frac{(2n)!}{2^n n!} \quad n = 0, 1, \dots \quad (2.75)$$

(ii) *Recurrence relation with respect to n .* Setting $x = 0$ either in the differentiation equation (2.50), or in the recurrence formula (2.58), yields for $0 \leq i \leq n$ (setting $\widehat{P}_{2i;2(i-1)}(0) := 0$),

$$4\widehat{P}_{2i;2(n+1)}(0) + \widehat{P}_{2i;2n}(0) - (n-i)(n+i-\frac{1}{2})\widehat{P}_{2i;2(n-1)}(0) = 0 \quad (2.76)$$

or equivalently from equation (2.74), $p_{i;n}$ satisfies the three term recurrence relation (setting $p_{i;i-1} := 0$),

$$p_{i;n+1} - p_{i;n} - 2(n-i)(2n+2i-1)p_{i;n-1} = 0. \quad (2.77)$$

With the boundary values (2.75), it follows by recurrence that for $m \geq 0$, $p_{i;i+m}$ is a polynomial in i of degree $[m/2]$ with positive integer coefficients, e.g.:

$$p_{i;i+1} = 1 \quad p_{i;i+2} = 8i + 3 \quad p_{i;i+3} = 24i + 15 \quad p_{i;i+4} = 192i^2 + 336i + 105. \quad (2.78)$$

(ii) *Recurrence relation with respect to both i and n .* Setting $x = 0$ in equation (2.67) yields for $0 \leq i \leq n-1$ (setting $\widehat{P}_{2(i+1);2i}(0) := 0$),

$$4\widehat{P}_{2(i+1);2n}(0) - 4\widehat{P}_{2i;2n}(0) - 2(n-i-1)\widehat{P}_{2(i+1);2(n-1)}(0) + (2i+2n-1)\widehat{P}_{2i;2(n-1)}(0) = 0 \quad (2.79)$$

or equivalently with equation (2.74) (setting $p_{i+1;i} := 0$),

$$2(2i+1)p_{i+1;n} - p_{i;n} + 4(2i+1)(n-i-1)p_{i+1;n-1} - (2i+2n-1)p_{i;n-1} = 0 \quad (2.80)$$

allowing also to get step by step $p_{i;n}$ for all i and n such that $0 \leq i \leq n$ starting from the boundary values (2.75).

Note that setting $x = 0$ in equation (2.44) (resp. (2.56)) allows to compute $\widehat{P}'_{i;n}(0)$ (resp. $\widehat{P}''_{i;n}(0)$), knowing $\widehat{P}_{i;n}(0)$ for several values of n .

2.7. Properties of the zeros

Several properties of the zeros of standard orthogonal polynomials have been known for a long time: e.g., as recalled in I-(3.3.7),

- (i) all the zeros are real, simple and located in the support $[a, b]$ of the measure;
- (ii) the zeros of polynomials with consecutive degrees separate each other.

We saw previously that Hermite SBO polynomials coincide in several special cases with standard orthogonal polynomials, and therefore share their properties, i.e. from equations (2.4), (2.22), (2.26) and (2.32),

$$\hat{P}_{0;0} = \hat{H}_0 = 1 \quad \hat{P}_{n;n} = \hat{P}_{n-1;n} = \hat{H}_n \quad n = 1, 2, \dots \quad (2.81)$$

$$\hat{P}_{0;n}(x) = 2^{-\frac{1}{2}n} \hat{H}_n(\sqrt{2}x) \quad \hat{P}_{1;2n+1}(x) = 2^{-3(n+\frac{1}{2})} \hat{H}_{2n+1}(\sqrt{2}x) \quad n = 0, 1, \dots \quad (2.82)$$

As a general result proven in section 3.3.7 of paper I:

- (i) for $0 \leq i \leq n$, $\hat{P}_{i;n}$ has at least m distinct real zeros of odd order, with $i \leq m \leq n$;
- (ii) $\hat{P}_{n-1;n}$, $n = 1, 2, \dots$ has n real and simple zeros.

It is recalled that the arguments used for standard orthogonal polynomials can no longer be used since the scalar product $(\cdot, \cdot)_0$, defined by equation I-(3.5) (see footnote 12), is not defined by a single integral. We observe that all special Hermite SBO polynomials of degree n listed in appendix E have numerically n real simple zeros. Nevertheless, we have not been able neither to prove this fact for all n , nor to build a counter-example.

Concerning the relative positions of the zeros of SBO polynomials with a given i and consecutive degrees, we only observe numerically that the zeros of the special Hermite SBO polynomials listed in appendix E separate each other. Once again, we have not been able to prove a general property.

3. Laguerre standard block orthogonal polynomials

The study use the same method and follows the same steps as for the Hermite polynomials in section 2. The (generalized) Laguerre polynomials $L_n^{(\alpha)}$, $n = 0, 1, \dots$, see appendix C, correspond to the measure defined by

$$a := 0 \quad b := \infty \quad w := e^{-x} x^\alpha \quad \alpha > -1. \quad (3.1)$$

3.1. Metric tensor components for a particular weight function w_2

(i) *Basic quantity* $\gamma_{j,k}$. With the same interval $[a, b]$, let us choose,

$$w_2 = w_2^{(\mu, \alpha)} := e^{-\mu x} x^\alpha \quad \alpha > -1 \quad \mu > 0. \quad (3.2)$$

This choice is motivated by the application considered in [7] for the special value $\mu = 2$. Our study will be also restricted in (ii) below to this special case, since it allows to push the calculation to its end analytically. Then, with the same steps as for equation (2.6), using the equations (B.6) and (B.3), one gets

$$\gamma_{j,k}(\alpha, \mu) := \left(L_j^{(\alpha)}, L_k^{(\alpha)} \right)_2 = \frac{\Gamma(\alpha + 1 + j)}{j!} \frac{\Gamma(\alpha + 1 + k)}{k!} \mu^{-(\alpha+1)} S(j, k, \alpha + 1; -\mu^{-1}, -\mu^{-1}) \quad (3.3)$$

where, the function S is defined in equation (C.8). Alternately, the result above can be obtained using the definite integral given in [5, 10]¹⁸. Another way to get the basic quantity $\gamma_{j,k}(\alpha, \mu)$ is to compute the scalar product $(F_s^{(\alpha)}, F_t^{(\alpha)})_2$ where, $F_z^{(\alpha)}(x)$ is the generating function of Laguerre polynomials given by equation (B.17). Then, $\gamma_{j,k}(\alpha, \mu)$ is the coefficient of $s^j t^k$ in the expansion of this scalar product, i.e.,

$$\left(F_s^{(\alpha)}, F_t^{(\alpha)} \right)_2 = \sum_{j,k=0}^{\infty} \gamma_{j,k}(\alpha, \mu) s^j t^k = \Gamma(\alpha + 1) (\mu + (\mu - 2)st - (\mu - 1)(s + t))^{-\alpha-1}. \quad (3.4)$$

For $\mu = 1$, i.e. for $w_2 = w$, using equation (C.14), one recovers the orthogonality relation of Laguerre polynomials, $\gamma_{j,k}(\alpha, 1) = h_j \delta_{j,k}$, with h_j given by equation (B.5). This result follows also from equation (3.4), using equation (C.1),

$$\left(F_s^{(\alpha)}, F_t^{(\alpha)} \right)_2 = \Gamma(\alpha + 1) (1 - st)^{-\alpha-1} = \Gamma(\alpha + 1) \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha-j}{j} (st)^j = \sum_{j=0}^{\infty} h_j (st)^j. \quad (3.5)$$

¹⁸ See, [5] 7.414 4. or [10] 2.19.14.6. p 477, using also the linear transformation formula [9] 15.3.6.

(ii) The special case $w_2 = w_2^{(2,\alpha)} = e^{-2x}x^\alpha$. As for Hermite polynomials, when $\mu = 2$ one gets simple expressions. From equations (3.3) and (C.15) setting $z = -\frac{1}{2}$, one finds

$$\gamma_{j,k}(\alpha, 2) = 2^{-(\alpha+1+j+k)} \frac{\Gamma(\alpha+1+j+k)}{j!k!} \quad (3.6)$$

i.e., once again, the same structure with a factor $\Gamma(c+j+k)$ multiplied by some factors depending separately on j and k . Alternately, the expression above follows also from equation (3.4),

$$\begin{aligned} \left(F_s^{(\alpha)}, F_t^{(\alpha)} \right)_2 &= \Gamma(\alpha+1)(2-s-t)^{-\alpha-1} \\ &= \Gamma(\alpha+1) \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^n \binom{-\alpha-1}{n} \binom{n}{m} 2^{-\alpha-1-n} s^m t^{n-m}. \end{aligned} \quad (3.7)$$

Then, setting $j = m$, $k = n - m$ and using equation (C.1), one recovers equation (3.6).

From now on, except explicitly noted, only this special case $\mu = 2$ will be considered.

Then, the corresponding polynomials $P_{i;n}$, $0 \leq i \leq n$ are called the *Laguerre SBO polynomials*. The special case $i = 1$ has already been considered in [7] with different choices of weight functions¹⁹. (See footnote 52 in appendix F for interrelations with our polynomial $\hat{P}_{i;n}$.)

3.2. Connection coefficients between block orthogonal and classical polynomials

As for Hermite polynomials, the relevant quantities are given in terms of determinants which can be evaluated using lemma D. One finds from equation I-(3.14), for $0 \leq i \leq n$,

$$Z_{i;n} = 2^{-(\alpha+i+n+1)(n-i+1)} \frac{\prod_{j=i}^n (j-i)! \Gamma(\alpha+1+i+j)}{\prod_{j=i}^n (j!)^2} \quad Z_{i;i-1} := 1 \quad (3.8)$$

and therefore, from equations I-(3.15), I-(3.7), I-(3.16) and I-(3.17) with k_n given by equation (B.2) [7]²⁰,

$$\hat{H}_{i;n} = 2^{-(\alpha+1+2n)} (n-i)! \Gamma(\alpha+1+i+n) \quad (3.9)$$

$$\hat{P}_{i;n} = \sum_{m=i}^n (-1)^m 2^{m-n} m! \binom{n-i}{m-i} \frac{\Gamma(\alpha+1+i+n)}{\Gamma(\alpha+1+i+m)} L_m^{(\alpha)} \quad (3.10)$$

$$L_n^{(\alpha)} = \sum_{m=i}^n (-1)^m 2^{m-n} \frac{1}{n!} \binom{n-i}{m-i} \frac{\Gamma(\alpha+1+i+n)}{\Gamma(\alpha+1+i+m)} \hat{P}_{i;m} \quad (3.11)$$

where, according to equation I-(3.13) and (B.2), for $i \geq 0$,

$$\hat{P}_{i;i} = \hat{L}_i^{(\alpha)} := (-1)^i i! L_i^{(\alpha)}. \quad (3.12)$$

The orthogonality and normalization of these polynomials with respect to the scalar product $(\cdot, \cdot)_2$ can be checked computing $\int_0^\infty dx e^{-2x} x^\alpha \hat{P}_{i;n}(x) \hat{P}_{i;n'}(x)$ with the expansion (3.10) and equations (3.6) and (C.14), thereby recovering equation (3.9).

¹⁹ See [7] section 2 where, to define the subspace \mathcal{P}_1^\perp , instead of considering the orthogonal polynomials Q_n with respect to (\cdot, \cdot) , the author computes numerically the “seed” polynomials $\tilde{Q}_n := x^n - (x^0, x^n)/(x^0, x^0)$, $n = 1, 2, \dots$, which readily fulfil the constraint $(x^0, \tilde{Q}_n) = 0$. This can be done easily because there is only one constraint, i.e. $i = 1$.

²⁰ Equations (3.9) (resp. (3.10)) generalizes the formula given in [7] (7) (resp. (A.1)) for the special case $i = 1$ with different choices of w and w_2 , see equations (F.16) and footnote 52.

Equations (3.10) and (3.11) should be compared to equations (B.6) and (B.9), respectively. More precisely, since $w = w_2^{(1,\alpha)}$, following arguments similar to those given for equation (2.30)²¹, it follows from equations (B.3) and (3.6) that,

$$\begin{aligned} (x^{i+j}, x^{i+k})^{(1,\alpha)} &= (x^j, x^k)^{(1,\alpha+2i)} = \Gamma(\alpha + 1 + 2i + j + k) \\ &= 2^{\alpha+1+2i} \left(2^j (i+j)! L_{i+j}^{(\alpha)}, 2^k (i+k)! L_{i+k}^{(\alpha)} \right)^{(2,\alpha)} \end{aligned} \quad (3.13)$$

where, $(,)^{(\mu,\alpha)}$ denotes the scalar products defined with the weigh function $w^{(\mu,\alpha)}$, given by equation (3.2). Thus, the metric tensor components of the scalar product $(,)^{(1,\alpha+2i)}$ in the basis $\{x^n, n = 0, 1, \dots\}$ and the metric tensor components of the scalar product $(,)^{(2,\alpha)}$ in the basis $\{2^n(i+n)! L_{i+n}^{(\alpha)}, n = 0, 1, \dots\}$ are equal apart from a common multiplicative factor. Hence, while G-SO with respect to $(,)^{(1,\alpha+2i)}$ of the basis $\{x^n, n = 0, 1, \dots\}$ defines the Laguerre polynomial $L_n^{(\alpha+2i)}$, G-SO with respect to $(,)^{(2,\alpha)}$ of the basis $\{2^n(i+n)! L_{i+n}^{(\alpha)}, n = 0, 1, \dots\}$ defines the polynomial $\hat{P}_{i,i+n}$. In other words, for given $i \geq 0$, the linear mapping defined by $x^n \mapsto 2^n(i+n)! L_{i+n}^{(\alpha)}$, $n = 0, 1, \dots$ is such that $L_n^{(\alpha+2i)} \mapsto \propto \hat{P}_{i,i+n}$. Therefore, apart from a multiplicative factor c , $\hat{P}_{i,i+n}$ reads as a linear combination of $2^m(i+m)! L_{i+m}^{(\alpha)}$, $m = 0, 1, \dots$ with *the same coefficients* as in the expansion of $L_n^{(\alpha+2i)}$ in terms of the monomials, i.e. from equation (B.6),

$$\hat{P}_{i,i+n} = c \sum_{m=0}^n \frac{(-1)^m}{n!} \frac{\Gamma(\alpha + 2i + 1 + n)}{\Gamma(\alpha + 2i + 1 + m)} \binom{n}{m} 2^m (i+m)! L_{i+m}^{(\alpha)}. \quad (3.14)$$

With $\hat{P}_{i,i+n}$ a monic polynomial, computing the coefficient of x^n fixes $c = (-1)^i 2^{-n} n!$. Changing $i+m$ into m and $i+n$ into n , with now $0 \leq i \leq n$, one recovers equation (3.10).

For $i = 0$, $\hat{P}_{0;n}$ has to be the unique standard monic polynomial $\hat{Q}_{2;n}$ corresponding to the weight function w_2 . Since, $w_2^{(2,\alpha)}(x) = 2^{-\alpha} w^{(1,\alpha)}(2x)$, with the normalization (B.2), one gets for $n \geq 0$,

$$\hat{P}_{0;n}(x) = 2^{-n} \hat{L}_n^{(\alpha)}(2x) = (-1)^n 2^{-n} n! L_n^{(\alpha)}(2x). \quad (3.15)$$

Thereby, setting $i = 0$ in equations (3.10) and (3.11) yields the remarkable linear relations between the two basic sets of polynomials $\{L_m^{(\alpha)}, m = 0, \dots, n\}$ and $\{L_m^{(\alpha)}(2x), m = 0, \dots, n\}$ for $n \geq 0$,

$$L_n^{(\alpha)}(2x) = (-1)^n \sum_{m=0}^n (-1)^m 2^m \frac{(\alpha + 1 + m)_{n-m}}{(n-m)!} L_m^{(\alpha)}(x) \quad (3.16)$$

$$L_n^{(\alpha)}(x) = 2^{-n} \sum_{m=0}^n \frac{(\alpha + 1 + m)_{n-m}}{(n-m)!} L_m^{(\alpha)}(2x). \quad (3.17)$$

These relations can be checked using the integral already quoted in footnote 18 and the orthogonality relation of Laguerre polynomials with h_n given by equation (B.5), e.g., one has

$$\int_0^\infty dx e^{-x} x^\alpha L_n^{(\alpha)}(2x) L_m^{(\alpha)}(x) = (-1)^{n+m} 2^m \frac{(\alpha + 1 + m)_{n-m}}{(n-m)!} h_m. \quad (3.18)$$

It follows from equations I-(3.10), I-(3.11), (3.10) and (B.7) that,

$$\hat{R}_{i;n} = -\frac{1}{2} (i(\alpha + i) + n(\alpha + n)) \quad (3.19)$$

$$\hat{S}_{i;n} = \frac{1}{8} \left(i^3 (2\alpha + i) + (2n^2 + 2(\alpha - 3)n - 3(\alpha - 1)) i(\alpha + i) + n(n-1)(\alpha + n)(\alpha + n - 1) \right). \quad (3.20)$$

²¹ What is new in the Laguerre case with respect to the Hermite one is that a shift of i in the powers of x can be interpreted as a shift in the parameter α . This is why in the Hermite case, this argumentation can be used with $i = 0$ only.

For general properties of SBO polynomials about the *projection operators*, the *integral representations* and the *normalizations*, see sections 3.3.1, 3.3.2 and 3.3.4 of paper I, respectively.

3.3. Differentiation formulae

For the scalar product (\cdot, \cdot) , the weight function w satisfies the differential equation (B.11), which implies the relations (B.12) and (B.13). Similarly, for the scalar product $(\cdot, \cdot)_2$ corresponding to the weight function $w_2^{(\mu, \alpha)}$ defined by equation (3.2), one has for any polynomials f and g ,

$$(xw_2^{(\mu, \alpha)})' = -(\mu x - \alpha - 1)w_2^{(\mu, \alpha)} \quad (3.21)$$

$$(f, xg')_2 = -(xf' - (\mu x - \alpha - 1)f, g)_2 \quad (3.22)$$

$$(xf'' - (\mu x - \alpha - 1)f', g)_2 = -(f', xg')_2 = (f, xg'' - (\mu x - \alpha - 1)g')_2. \quad (3.23)$$

(i) For $0 \leq \ell < i - 1$, the polynomial $xL_\ell^{\alpha'}$ of degree ℓ is in the subspace \mathcal{P}_i . Therefore, $xL_\ell^{\alpha'}$ and $\hat{P}_{i;n}$ for $i \leq n$ are orthogonal with respect to (\cdot, \cdot) . From equation (B.12), one has for $0 \leq \ell \leq i - 1$ and $0 \leq i \leq n$,

$$0 = (\hat{P}_{i;n}, xL_\ell^{(\alpha)'}) = -\left(x\hat{P}_{i;n}' - (x - \alpha - 1)\hat{P}_{i;n}, L_\ell^{(\alpha)}\right) = -\left(x\hat{P}_{i;n}' - x\hat{P}_{i;n}, L_\ell^{(\alpha)}\right). \quad (3.24)$$

Hence, the particular polynomial $x\hat{P}_{i;n}' - x\hat{P}_{i;n}$ of degree $n+1$ is orthogonal to the subspace \mathcal{P}_i with respect to (\cdot, \cdot) , and thus reads $\sum_{j=i}^{n+1} \alpha_{i;n,j} \hat{P}_{i;j}$. Computing the coefficient of x^{n+1} in $x\hat{P}_{i;n}' - x\hat{P}_{i;n}$ fixes $\alpha_{i;n,n+1} = -1$. Now, with respect to the scalar product $(\cdot, \cdot)_2$, one has from equations (1.2) and (3.22), for $i \leq j \leq n+1$,

$$\begin{aligned} \alpha_{i;n,j} \hat{H}_{i;j} &= (\hat{P}_{i;j}, x\hat{P}_{i;n}' - x\hat{P}_{i;n})_2 = (\hat{P}_{i;j}, x\hat{P}_{i;n}')_2 - (x\hat{P}_{i;j}, \hat{P}_{i;n})_2 \\ &= -(x\hat{P}_{i;j}' - (\mu - 1)x\hat{P}_{i;j}, \hat{P}_{i;n})_2 - (\alpha + 1)\hat{H}_{i;n}\delta_{j,n}. \end{aligned} \quad (3.25)$$

In the special case $\mu = 2$, it turns out that $x\hat{P}_{i;j}' - (\mu - 1)x\hat{P}_{i;j}$ is the particular polynomial $x\hat{P}_{i;j}' - x\hat{P}_{i;j}$, introduced above for $j = n$ (i.e. of degree $j+1$ in the subspace \mathcal{P}_i^\perp), and which reads $\sum_{k=i}^{j+1} \alpha_{i;j,k} \hat{P}_{i;k}$. Thus, for $\mu = 2$, *considered henceforth*, one has

$$\alpha_{i;n,j} \hat{H}_{i;j} = -\left(\alpha_{i;j,n} \sum_{k=i}^{j+1} \delta_{k,n} + (\alpha + 1)\delta_{j,n}\right) \hat{H}_{i;n}. \quad (3.26)$$

It follows that for $j = i, \dots, n-2$, $\alpha_{i;n,j}$ vanishes. It remains to consider the cases $j = n-1$ and $j = n$. With $\alpha_{i;n-1,n} = -1$ and equation (3.9), one gets

$$\kappa_{i;n}(\alpha) := \alpha_{i;n,n-1} = \frac{\hat{H}_{i;n}}{\hat{H}_{i;n-1}} = \frac{1}{4}(n-i)(\alpha + i + n) \quad \alpha_{i;n,n} = -\frac{1}{2}(\alpha + 1). \quad (3.27)$$

Finally, for $\mu = 2$ and $0 \leq i \leq n$, one finds the *first order differentiation formula* [7]²²,

$$x\hat{P}_{i;n}' = -\hat{P}_{i;n+1} + \left(x - \frac{1}{2}(\alpha + 1)\right)\hat{P}_{i;n} + \kappa_{i;n}(\alpha)\hat{P}_{i;n-1} \quad (3.28)$$

which still holds for $n = i$, since then $\kappa_{i;n}(\alpha)$ vanishes. This equation has to be compared with the differentiation formula (B.16). For $i = 0$, using equation (3.15), one gets

$$2xL_n^{(\alpha)'} = (n+1)L_{n+1}^{(\alpha)} + (x - \alpha - 1)L_n^{(\alpha)} - (n + \alpha)L_{n-1}^{(\alpha)} \quad (3.29)$$

²² This generalizes the formula given in [7] (3) for the special case $i = 1$ with different choices of w and w_2 , see footnote 52.

which follows from, e.g., adding the two differentiation formulae (B.14) and (B.16).

With the boundary value of $\widehat{P}_{i,i}$ given in terms of Laguerre polynomials by equation (3.12), the differentiation formula (3.28) allows to get step by step $\widehat{P}_{i,n}$ for all i and n with $0 \leq i \leq n$. A table of these polynomials for $i = 0, 1, 2$ and $n = i, i+1, \dots, i+4$ is given in appendix F. In particular, setting $n = i+1$ and using equations (3.12), (B.2) and (B.16), one finds for $n = i \geq 0$,

$$\widehat{P}_{i,i+1} = \widehat{L}_{i+1}^{(\alpha)} + \frac{1}{2}(\alpha + 1 + 2i) \widehat{L}_i^{(\alpha)} = (-1)^{i+1}(i+1)! L_{i+1}^{(\alpha)} + (-1)^i \frac{1}{2} i! (\alpha + 1 + 2i) L_i^{(\alpha)}. \quad (3.30)$$

This result can be obtained directly setting $n = i+1$ in equation (3.10).

(ii) Following the same method again, for $0 \leq \ell < i \leq n$, the polynomial $xL_\ell^{\alpha''} - (x - \alpha - 1)L_\ell^{\alpha'}$ of degree ℓ is orthogonal to $\widehat{P}_{i,n}$ with respect to the scalar product (\cdot, \cdot) , and from equation (B.13), one has

$$0 = \left(\widehat{P}_{i,n}, xL_\ell^{\alpha''} - (x - \alpha - 1)L_\ell^{\alpha'} \right) = \left(x\widehat{P}_{i,n}'' - (x - \alpha - 1)\widehat{P}_{i,n}', L_\ell^{(\alpha)} \right). \quad (3.31)$$

Hence, the particular polynomial $x\widehat{P}_{i,n}'' - (x - \alpha - 1)\widehat{P}_{i,n}'$ of degree n belongs to \mathcal{P}_i^\perp and reads $\sum_{j=i}^n \gamma_{i,n,j} \widehat{P}_{i,j}$. Computing the coefficient of x^n in $x\widehat{P}_{i,n}'' - (x - \alpha - 1)\widehat{P}_{i,n}'$ fixes $\gamma_{i,n,n} = -n$. With the scalar product $(\cdot, \cdot)_2$, one has from equations (1.2), (3.22) and (3.23), for $i \leq j \leq n$,

$$\begin{aligned} \gamma_{i,n,j} \widehat{H}_{i,j} &= (\widehat{P}_{i,j}, x\widehat{P}_{i,n}'' - (x - \alpha - 1)\widehat{P}_{i,n}')_2 \\ &= (\widehat{P}_{i,j}, x\widehat{P}_{i,n}'' - (\mu x - \alpha - 1)\widehat{P}_{i,n}')_2 + (\mu - 1)(\widehat{P}_{i,j}, x\widehat{P}_{i,n}')_2 \\ &= \left(x\widehat{P}_{i,j}'' - ((\mu - 1)x - \alpha - 1)\widehat{P}_{i,j}', \widehat{P}_{i,n} \right)_2 - \mu \left(x\widehat{P}_{i,j}' - (\mu - 1)x\widehat{P}_{i,j}, \widehat{P}_{i,n} \right)_2 \\ &\quad - (\mu - 1)(\alpha + 1)\widehat{H}_{i,n}\delta_{j,n}. \end{aligned} \quad (3.32)$$

In the special case $\mu = 2$, once again, $x\widehat{P}_{i,j}'' - ((\mu - 1)x - \alpha - 1)\widehat{P}_{i,j}'$ (resp. $x\widehat{P}_{i,j}' - (\mu - 1)x\widehat{P}_{i,j}$) is the particular polynomial $x\widehat{P}_{i,j}'' - (x - \alpha - 1)\widehat{P}_{i,j}'$ (resp. $x\widehat{P}_{i,j}' - x\widehat{P}_{i,j}$), introduced above for $j = n$ (i.e. of degree j (resp. $j+1$) in the subspace \mathcal{P}_i^\perp), and which reads $\sum_{k=i}^j \gamma_{i,j,k} \widehat{P}_{i,k}$ (resp. $\sum_{k=i}^{j+1} \alpha_{i,j,k} \widehat{P}_{i,k}$). Thus, one has

$$\gamma_{i,n,j} \widehat{H}_{i,j} = \left(\gamma_{i,j,n} \sum_{k=i}^j \delta_{k,n} - 2\alpha_{i,j,n} \sum_{k=i}^{j+1} \delta_{k,n} - (\alpha + 1)\delta_{j,n} \right) \widehat{H}_{i,n}. \quad (3.33)$$

For $n = i$, and therefore $j = i$, it is already known that $\gamma_{i,i,i} = -i$. For $n \geq i+1$, it follows from above that for $n \geq i+2$ and $j = i, \dots, n-2$, $\gamma_{i,n,j}$ vanishes. It remains to consider the case $j = n-1$. From $\alpha_{i,n-1,n} = -1$ and equation (3.27), one gets

$$\gamma_{i,n,n-1} = -2\alpha_{i,n-1,n} \frac{\widehat{H}_{i,n}}{\widehat{H}_{i,n-1}} = 2\kappa_{i,n}(\alpha). \quad (3.34)$$

Finally, in the case $\mu = 2$, one finds the *second order differentiation formula* for $0 \leq i \leq n$ [7]²³,

$$x\widehat{P}_{i,n}'' + (\alpha + 1 - x)\widehat{P}_{i,n}' + n\widehat{P}_{i,n} = 2\kappa_{i,n}(\alpha)\widehat{P}_{i,n-1} \quad (3.35)$$

where, $\kappa_{i,n}(\alpha)$ is defined by equation (3.27). For $n = i$, the relation still holds, since then $\kappa_{i,n}(\alpha)$ vanishes, and using equation (3.12), the result coincides with the differential equation (B.15). For $i = 0$, using equation (3.15), the relation follows, e.g., from equations (B.14) and (B.15).

3.4. Recurrence relation with respect to the degree n

²³ This generalizes the formula given in [7] (6) for the special case $i = 1$ with different choices of w and w_2 , see footnote 52.

Taking the derivative of the first order differentiation formula (3.28), and eliminating $x\hat{P}_{i;n}''$ with the second order differentiation formula (3.35) yield

$$-\hat{P}_{i;n+1}' + \frac{1}{2}(\alpha - 1)\hat{P}_{i;n}' + (n + 1)\hat{P}_{i;n}' + \frac{1}{4}(n - i)(\alpha + 1 + i + n)(\hat{P}_{i;n-1}' - 2\hat{P}_{i;n-1}) = 0. \quad (3.36)$$

Then, multiplying both sides of this equation by x and using again equation (3.28) to eliminate the first order derivatives $\hat{P}_{i;n+1}', \hat{P}_{i;n}'$ ²⁴ and $\hat{P}_{i;n-1}'$ give the *five term linear recurrence formula* for $0 \leq i \leq n$,

$$\begin{aligned} & \hat{P}_{i;n+2} - (x - 1)\hat{P}_{i;n+1} + \left(\frac{1}{2}(\alpha + 1 + 2n)x - 2\kappa_{i;n}(\alpha) - \frac{1}{4}(\alpha^2 + \alpha + 2n) \right) \hat{P}_{i;n} \\ & - \kappa_{i;n}(\alpha)(x + 1)\hat{P}_{i;n-1} - \kappa_{i;n}(\alpha)\kappa_{i;n-1}(\alpha)\hat{P}_{i;n-2} = 0 \end{aligned} \quad (3.37)$$

where, $\kappa_{i;n}(\alpha)$ is defined by equation (3.27). A direct calculation shows that this relation still holds for $n = i + 1$ and $n = i$, where it reduces to a four and three term recurrence formula, respectively. For $n = i$, using equation (3.30), one finds for $i \geq 0$,

$$\hat{P}_{i;i+2} = (x - 1)\hat{L}_{i+1}^{(\alpha)} + \frac{1}{4}(\alpha^2 - \alpha - 2 - 2i)\hat{L}_i^{(\alpha)}. \quad (3.38)$$

As the differentiation formula (3.28), the recurrence formula (3.37) can be used to get step by step $\hat{P}_{i;n}$ for all i and n with $0 \leq i \leq n$ starting from $\hat{P}_{i;i}$ and $\hat{P}_{i;i+1}$ given by equations (3.12) and (3.30), respectively (see appendix F).

3.5. Recurrence relations with respect to both i and n

(i) Setting $j = i + 1$ in equation I-(3.58) and using equations (3.10) and (3.11), one can express $\hat{P}_{i+1;n}$ in terms of the polynomials $\hat{P}_{i;\ell}$ with $i \leq \ell \leq n$. The calculation is very similar to what was done for Hermite polynomials in section 2.5. One finds

$$\hat{P}_{i+1;n} = \sum_{\ell=i}^n \hat{P}_{i;\ell} \frac{2^{\ell-n}}{n-i} \binom{n-i}{n-\ell} \frac{\Gamma(\alpha + 2 + i + n)}{\Gamma(\alpha + 1 + i + \ell)} \sum_{m=\sup(i+1,\ell)}^n (-1)^{m-\ell} \binom{n-\ell}{m-\ell} \frac{m-i}{\alpha + 1 + i + m} \quad (3.39)$$

and using equations (2.61) and (2.62) (with $p = n - \ell$, $\rho = \ell - i$ and $\sigma = \alpha + 1 + i + \ell > 0$) to compute the sum over m yield for $0 \leq i < n$,

$$\hat{P}_{i+1;n} = \hat{P}_{i;n} - 2^{-n}(\alpha + 1 + 2i)(n - i - 1)! \sum_{\ell=i}^{n-1} \frac{2^\ell}{(\ell - i)!} \hat{P}_{i;\ell}. \quad (3.40)$$

This recurrence relation provide another way of computing step by step $\hat{P}_{i;n}$ for all i and n with $0 \leq i \leq n$, starting with the boundary values $\hat{P}_{0;n}$ given by equation (3.15) (see appendix F). Thus, for $i = 0$, one gets for $n \geq 1$,

$$\hat{P}_{1;n} = (-1)^n 2^{-n} L_n^{(\alpha)}(2x) - 2^{-n}(\alpha + 1)(n - 1)! \sum_{\ell=0}^{n-1} (-1)^\ell L_\ell^{(\alpha)}(2x) \quad (3.41)$$

i.e. an expansion in terms of Laguerre polynomials of the variable $2x$, instead of x as given by equation (3.10).

(ii) On the right-hand side of equation (3.40), the sum over ℓ depends on n through the upper bound $n - 1$ of the sum only. Therefore, replacing n by $n - 1$ in this equation and eliminating $\sum_{\ell=i}^{n-2} \dots$ between the equation obtained thereby and equation (3.40) yield for $0 \leq i \leq n - 1$,

$$2\hat{P}_{i+1;n} - 2\hat{P}_{i;n} - (n - i - 1)\hat{P}_{i+1;n-1} + (\alpha + i + n)\hat{P}_{i;n-1} = 0. \quad (3.42)$$

²⁴ Keeping the term $\hat{P}_{i;n}'$ provides another first order differentiation formula.

This relation still holds for $n = i + 1$, where it reduces to a three term recurrence recovering equation (3.40).

This *four term linear recurrence relation* provides an easy way of computing step by step $\hat{P}_{i;n}$ for all i and n with $0 \leq i \leq n$, starting from the boundary values of $\hat{P}_{i;i}$ given by equation (3.12) (see appendix F).

3.6. Special value at $x = 0$

From equations (3.10), (B.8) and (C.1), one finds for $0 \leq i \leq n$,

$$\hat{P}_{i;n}(0) = (-1)^i 2^{-(n-i)} (\alpha + 1)_i (\alpha + 1 + 2i)_{n-i} F(-(n-i), \alpha + 1 + i; \alpha + 1 + 2i; 2) \quad (3.43)$$

where, F is a terminating hypergeometric function, for which we don't know an explicit expression for all i and n . These values can be computed by recurrence as follows.

(i) *Boundary values for $n = i$ and $i = 0$.* From equations (3.12), (3.15) and (B.8), one gets

$$\hat{P}_{i;i}(0) = (-1)^i i! L_i^{(\alpha)}(0) = (-1)^i (\alpha + 1)_i \quad i = 0, 1, \dots \quad (3.44)$$

$$\hat{P}_{0;n}(0) = (-1)^n 2^n n! L_n^{(\alpha)}(0) = (-1)^n 2^n (\alpha + 1)_n \quad n = 0, 1, \dots \quad (3.45)$$

Setting for $0 \leq i \leq n$ ²⁵,

$$\hat{P}_{i;n}(0) := (-1)^n 2^{-(n-i)} (\alpha + 1)_i p_{i;n}^{(\alpha)} \quad (3.46)$$

one finds the boundary values,

$$p_{i;i}^{(\alpha)} = 1 \quad i = 0, 1, \dots \quad p_{0;n}^{(\alpha)} = (\alpha + 1)_n \quad n = 0, 1, \dots \quad (3.47)$$

(ii) *Recurrence relation with respect to n .* Setting $x = 0$ in the differentiation equation (3.28) yields for $0 \leq i \leq n$ (setting $\hat{P}_{i;i-1}(0) := 0$),

$$4\hat{P}_{i;n+1}(0) + 2(\alpha + 1)\hat{P}_{i;n}(0) - (n - i)(\alpha + i + n)\hat{P}_{i;n-1}(0) = 0 \quad (3.48)$$

or equivalently from equation (3.46), $p_{i;n}^{(\alpha)}$ satisfies the three term recurrence relation (setting $p_{i;i-1}^{(\alpha)} := 0$),

$$p_{i;n+1}^{(\alpha)} - (\alpha + 1)p_{i;n}^{(\alpha)} - (n - i)(\alpha + i + n)p_{i;n-1}^{(\alpha)} = 0. \quad (3.49)$$

With the boundary values (3.47), it follows by recurrence that for $m \geq 0$, $p_{i;i+m}^{(\alpha)}$ is a monic polynomial in α of degree m with positive integer coefficients, and also a polynomial in i of degree $\lfloor m/2 \rfloor$, e.g.:

$$\begin{aligned} p_{i;i+1}^{(\alpha)} &= \alpha + 1 & p_{i;i+2}^{(\alpha)} &= \alpha^2 + 3\alpha + 2(1 + i) & p_{i;i+3}^{(\alpha)} &= (\alpha + 1)(\alpha^2 + 5\alpha + 6(1 + i)) \\ p_{i;i+4}^{(\alpha)} &= \alpha^4 + 10\alpha^3 + (35 + 12i)\alpha^2 + 2(25 + 18i)\alpha + 12(1 + i)(2 + i). \end{aligned} \quad (3.50)$$

(iii) *Recurrence relation with respect to both i and n .* Setting $x = 0$ in equation (3.42) yields for $0 \leq i \leq n - 1$ (setting $\hat{P}_{i+1;i}(0) := 0$),

$$2\hat{P}_{i+1;n}(0) - 2\hat{P}_{i;n}(0) - (n - i - 1)\hat{P}_{i+1;n-1}(0) + (\alpha + i + n)\hat{P}_{i;n-1}(0) = 0 \quad (3.51)$$

or equivalently with equation (3.46),

$$2(2\alpha + 1 + i)p_{i+1;n}^{(\alpha)} - p_{i;n}^{(\alpha)} + 2(\alpha + 1 + i)(n - i - 1)p_{i+1;n-1}^{(\alpha)} - (\alpha + i + n)p_{i;n-1}^{(\alpha)} = 0 \quad (3.52)$$

²⁵ See equation (3.56) below, $p_{i;n}$ defined for Hermite polynomials by equation (2.74) and $p_{i;n}^{(\alpha)}$ are related by $p_{i;n} = 2^{n-i} p_{i;n}^{(-\frac{1}{2})}$.

allowing also to get step by step $p_{i,n}^{(\alpha)}$ for all i and n such that $0 \leq i \leq n$ starting from the boundary values (3.47).

Note that setting $x = 0$ in equation (3.35) (resp. its derivative with respect to x) allows to compute $\hat{P}'_{i,n}(0)$ (resp. $\hat{P}''_{i,n}(0)$), knowing $\hat{P}_{i,n}(0)$ for several values of n .

3.7. Properties of the zeros

From equations (3.12) and (3.15), the particular Laguerre SBO polynomials,

$$\hat{P}_{n;n} = \hat{L}_n^{(\alpha)} \quad \hat{P}_{0;n}(x) = 2^{-n} \hat{L}_n^{(\alpha)}(2x) \quad n = 0, 1, \dots \quad (3.53)$$

share the properties, especially of the zeros, of the standard orthogonal polynomials.

It has been shown in section 3.3.7 of paper I that:

- for $0 \leq i \leq n$, $P_{i;n}$ has at least m distinct non-negative real zeros of odd order, with $i \leq m \leq n$;
- for $n \geq 1$, $P_{n-1;n}$ has n non-negative real and simple zeros.

We observe that all the special Laguerre SBO polynomials of degree n listed in appendix F have numerically n real non-negative simple zeros. Nevertheless, we have not been able neither to prove this fact in general, nor to build a counter-example.

Concerning the relative positions of the zeros of SBO polynomials with a given i and consecutive degrees, in the Laguerre case one readily finds a counter-example, e.g., with $\alpha = 0$, from equations (F.6) and (F.7), the zeros read

$$\hat{P}_{1;2} = x^2 - \frac{5}{2}x + \frac{1}{2} \quad x_1 = \frac{5 - \sqrt{17}}{4} \approx 0.219 \quad x_2 = \frac{5 + \sqrt{17}}{4} \approx 2.281 \quad (3.54)$$

$$\hat{P}_{1;3} = x^3 - 5x^2 + 5x - 1 \quad y_1 = 2 - \sqrt{3} \approx 0.268 \quad y_2 = 1 \quad y_3 = 2 + \sqrt{3} \approx 3.732 \quad (3.55)$$

where, $x_1 < y_1 < y_2 < x_2 < y_3$, and therefore the zeros do not separate each other, as for standard orthogonal polynomials. Once again, we have not been able to prove a general property.

3.8. Interrelations between Hermite and Laguerre cases

The interrelations between Hermite and Laguerre polynomials are recalled in appendix B (item 6). Using the relation (B.19) (resp. (B.20)), the expansion (2.20) (resp. (2.21)) for the even (resp. odd) polynomials in the Hermite case, and the expansion (3.10) for the Laguerre case, setting $\alpha = -\frac{1}{2}$ (resp. $\frac{1}{2}$), one finds for $0 \leq i \leq n$,

$$\hat{P}_{2i-1;2n}^{(\text{Hermite})}(x) = \hat{P}_{2i;2n}^{(\text{Hermite})}(x) = \hat{P}_{i;n}^{(\text{Laguerre } \alpha=-\frac{1}{2})}(x^2) \quad (3.56)$$

$$\hat{P}_{2i;2n+1}^{(\text{Hermite})}(x) = \hat{P}_{2i+1;2n+1}^{(\text{Hermite})}(x) = x \hat{P}_{i;n}^{(\text{Laguerre } \alpha=\frac{1}{2})}(x^2) \quad (3.57)$$

i.e. the same interrelations as between Hermite and Laguerre polynomials. Let us recall that, while the choice of i determines the subspace \mathcal{P}_i of polynomials of degree at most $i-1$, the complementary and orthogonal subspace \mathcal{P}_i^\perp , equation (1.3), depends on the choice of the scalar product (\cdot, \cdot) . Therefore, $\mathcal{P}_i^{(\text{Hermite})\perp}$, $\mathcal{P}_i^{(\text{Laguerre } \alpha=-\frac{1}{2})\perp}$ and $\mathcal{P}_i^{(\text{Laguerre } \alpha=\frac{1}{2})\perp}$ are, a priori, different subspaces.

Relations (3.56) and (3.57) allow a cross check of many similar formulae obtained in sections 2 and 3, and also of the explicit expressions of the Hermite and the Laguerre SBO polynomials listed in appendices E and F, respectively.

Conclusion

In a first paper [1] (paper I), we have defined general *block orthogonal polynomials*. Their properties have been investigated, pointing out what can be generalized and ascribed to Euclidean vector space, endowed with two distinct scalar products, and to polynomial algebra. A general study of the special case of *standard block orthogonal* (SBO) *polynomials*, has been made, emphasizing the similarities to and the differences from

the study of the *standard orthogonal polynomials*. For a given $i = 0, 1, \dots$, these real polynomials of degree n , $P_{i;n}$, $n = i, i + 1, \dots$, are orthogonal to the i -dimensional subspace \mathcal{P}_i of polynomials of degree less than i with respect to a first Euclidean scalar product (\cdot, \cdot) . In addition, they are mutually orthogonal with respect to a second Euclidean scalar product $(\cdot, \cdot)_2$. They form an orthogonal basis with respect to $(\cdot, \cdot)_2$ of the subspace \mathcal{P}_i^\perp of codimension i , orthogonal to \mathcal{P}_i with respect to (\cdot, \cdot) , and complementary to \mathcal{P}_i with respect to the space of polynomials of any degree. Based on the Gram-Schmidt orthogonalization (G-SO), we have determined these polynomials uniquely, apart from a constant multiplicative factor for each polynomial, in terms of determinants whose entries are the metric tensor components of both scalar products considered.

In this second paper, we apply the general results obtained in paper I, assuming the first scalar product (\cdot, \cdot) corresponds to the Hermite (resp. Laguerre) *classical polynomials*. In both cases, it turns out that for a particular choice of the second scalar product $(\cdot, \cdot)_2$, we get explicitly the connection coefficients between the Hermite (resp. Laguerre) polynomials, and what we call the *Hermite* (resp. *Laguerre*) *SBO polynomials*. These polynomials satisfy all the general properties derived in section 3.3 of paper I for the projection operators, the integral representations and the normalizations. Furthermore, we establish recurrence relations with respect to n and i , and differentiation formulae, allowing a step by step determination of these SBO polynomials. As for the classical orthogonal polynomials, these relations are based on differential equations satisfied by the weight functions $w(x)$ and $w_2(x)$ defining, respectively, the two scalar products. Partial properties for the zeros of these polynomials are also obtained. We point out the differences with the classical orthogonal polynomials, basically due to the fact that two, instead of one, distinct scalar products are involved. As the Hermite and the Laguerre polynomials, the Hermite and the Laguerre SBO polynomials can be related together. In the Hermite case, the measures considered being symmetric with respect to the origin, additional parity properties occur.

This paper recovers and improves results already obtained by Giraud *et al* [6–8] for $i = 1$, and partly for $i = 2$. In particular, we get new recurrence and differentiation formulae. We generalize these results making a systematic study for all non-negative integer values of i . As recalled in paper I, $P_{i;n}$, $n = i, i + 1, \dots$ provide a basis of the subspace \mathcal{P}_i^\perp , suitable for applications requiring to take into account i linearly independent constraints which can be recast into an orthogonality relation with the subspace \mathcal{P}_i with respect to (\cdot, \cdot) . (We have shown in paper I that an extension of similar methods to constraints associated with different scalar products is not straightforward, and may have no solution, a unique solution or an infinite number of solutions.) These constrained polynomials have been first considered in [6–8] with $i = 1$ to implement the matter conservation in view of application of the Hohenberg-Kohn functional in some problems of nuclear physics.

Open questions still remain. Why can we perform analytical calculations only for very special choices of the weight function w_2 (precisely those made in the applications above), once w is chosen as the classical Hermite (resp. Laguerre) weight function? Is there any ‘integrability’ structure to discover? Nevertheless, let us recall that for any proper weight functions, determination of the SBO polynomials is a well-defined problem. G-SO is well-suited for numerical calculations and it is not necessary for applications to have analytical expressions. We have not been able to get closed formulae for the connection coefficients between $P_{i;n}$ and x^m (we observe in all the special values of $P_{i;n}$ given in appendices E and F, alternate signs in the expansion in powers of x). Can we establish that, both in Hermite and Laguerre cases, $P_{i;n}$ has n distinct real zeros? Similarly, do the zeros of the Hermite SBO polynomials with a given i and consecutive degrees separate each other? (We establish this is not true for the Laguerre case.)

As a byproduct, for completeness and to underline the similarities to and the differences from the classical polynomials, we recover in appendices well-known properties of Hermite and Laguerre polynomials in a unified way (including the expansions of monomials in terms of classical polynomials, not given in the standard textbooks). In addition, also reported in appendix, we have been led to consider several terminating generalized hypergeometric series with one or more variables often set to unity.

The Gegenbauer and Jacobi polynomials will be similarly considered in a forthcoming paper.

We have already underlined in the conclusion of paper I (see footnote 44 of paper I), that the general formulae established therein can be used to define block orthogonal polynomials for any given i -dimensional subspace \mathcal{E}_i , possibly different from \mathcal{P}_i (see footnote 51). Then the block orthogonal polynomials would not be the standard ones and require a new similar study.

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Appendix A. Hermite polynomials

It is shown in appendix I-A that all the relevant quantities $(Z_n, h_n, Q_n, a_{m,n}, b_{m,n}, \dots)$ can be expressed in terms of determinants with the metric tensor components as elements, except possibly for the last row,

$$g_{j,k} := (x^j, x^k) = c_{j+k} := \int_a^b dx w(x) x^{j+k}. \quad (A.1)$$

These general results are now applied to the *classical orthogonal polynomials*, i.e. for particular choices of the measure defining the scalar product. Thereby, well-known results are recovered using a unique method based on the calculation of some special determinants with gamma functions as elements (see lemma D in appendix D) and also the evaluation of some finite sums involving binomial coefficients and ratios of gamma functions (see lemmas C.1 and C.2 in appendix C). In addition, since the weight functions w considered satisfy some differential equations, integrations by part with vanishing boundary conditions yield some relations between scalar products (*this is still valid for particular SBO polynomial*, see sections 2.3 and 3.3.).

(1) *Definition.* Hermite polynomials H_n , $n = 0, 1, \dots$, are defined with the measure and normalization ²⁶,

$$a := -\infty \quad b := \infty \quad w := e^{-x^2} \quad (A.2)$$

$$k_n := 2^n \quad n = 0, 1, \dots \quad (A.3)$$

The measure being symmetric with respect to the origin, $H_n(-x) = (-1)^n H_n(x)$.

(2) *Connection coefficients.* From equation (A.1) and the *Euler integral* ²⁷, the nonzero moments read

$$c_{2j} = \int_{-\infty}^{\infty} dx e^{-x^2} x^{2j} = \Gamma(\frac{1}{2} + j) \quad j = 0, 1, \dots \quad (A.4)$$

Therefore, all the determinants which occur in equations I-(A.17)–I-(A.28) can be obtained with lemma D and the duplication formula (see footnote 6). For $n = 0, 1, \dots$ and $m = 0, \dots, n$, one finds ²⁸,

$$Z_n^{(e)} = \prod_{j=0}^n j! \Gamma(\frac{1}{2} + j) \quad Z_n^{(o)} = \prod_{j=0}^n j! \Gamma(\frac{3}{2} + j) \quad (A.5)$$

$$h_n = 2^n n! \pi^{\frac{1}{2}} \quad (A.6)$$

$$a_{2m,2n} = k_{2n} (-1)^{n-m} \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(\frac{1}{2} + m)} \binom{n}{m} = (-1)^{n-m} 2^{2m} \frac{(2n)!}{(2m)! (n-m)!} \quad (A.7)$$

$$a_{2m+1,2n+1} = k_{2n+1} (-1)^{n-m} \frac{\Gamma(\frac{3}{2} + n)}{\Gamma(\frac{3}{2} + m)} \binom{n}{m} = (-1)^{n-m} 2^{2m+1} \frac{(2n+1)!}{(2m+1)! (n-m)!} \quad (A.8)$$

$$H_n = \sum_{m=0}^{[n/2]} (-1)^m \frac{n!}{m! (n-2m)!} (2x)^{n-2m} \quad (A.9)$$

$$\hat{r}_n = 0 \quad \hat{s}_n = -\frac{1}{4} n(n-1) \quad (A.10)$$

$$H_{2n}(0) = (-1)^n 2^{2n} \left(\frac{1}{2}\right)_n = (-1)^n \frac{(2n)!}{n!} \quad H_{2n+1}(0) = 0. \quad (A.11)$$

²⁶ See, e.g., [4] 10.13 (1) and (4).

²⁷ See, e.g., [3] 6.1.1.

²⁸ See, e.g., [4] 10.13 (4) for h_n , (9) for the expansion (A.9), (4) for r_n only (see footnote 47 in paper I), (15) for $H_n(0)$ and (5) for the three term recurrence formula.

Similarly, the connection coefficient $b_{m,n}$, not given in the standard textbook already quoted, can be obtained from equations I-(A.10), I-(A.22), I-(A.28), (D.2), (C.3) and the duplication formula (see footnote 6). For $m = 0, \dots, n$ one has ²⁹,

$$\begin{aligned} b_{2m,2n} &= \frac{k_{2m}^{-1}}{m!} \sum_{\ell=0}^m (-1)^{m-\ell} \binom{m}{\ell} \frac{\Gamma(\frac{1}{2} + n + \ell)}{\Gamma(\frac{1}{2} + \ell)} \\ &= 2^{-2m} \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(\frac{1}{2} + m)} \binom{n}{m} = 2^{-2n} \frac{(2n)!}{(2m)!(n-m)!} \end{aligned} \quad (\text{A.12})$$

$$b_{2m+1,2n+1} = 2^{-(2m+1)} \frac{\Gamma(\frac{3}{2} + n)}{\Gamma(\frac{3}{2} + m)} \binom{n}{m} = 2^{-(2n+1)} \frac{(2n+1)!}{(2m+1)!(n-m)!} \quad (\text{A.13})$$

$$(2x)^n = \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m)!} H_{n-2m}. \quad (\text{A.14})$$

These expressions of $b_{m,n}$ can also be obtained directly with the following lemma, setting $\rho_k = (-)^k 2^{2k} \Gamma(\frac{1}{2} + k)$ and $\sigma_j = \Gamma(\frac{1}{2} + j)$.

Lemma A. *With ρ_j and σ_j some non-zero complex numbers, the matrices \mathbf{A} and \mathbf{B} defined by, $A_{j,k} := (-1)^j \binom{k}{j} \rho_k / \sigma_j$ and $B_{j,k} := (-1)^j \binom{k}{j} \sigma_k / \rho_j$ for $j, k = 0, 1, \dots$, are inverse one from the other.*

Proof. Since the binomial coefficient $\binom{k}{j}$ vanishes if $k < j$, the matrices \mathbf{A} and \mathbf{B} are triangular with zeros under the diagonal and nonzero diagonal elements. Then, this known lemma ³⁰, is a direct consequence of the binomial formula. Indeed, \mathbf{AB} is also a triangular matrix and one has

$$(\mathbf{AB})_{j,k} = \frac{\sigma_k}{\sigma_j} \sum_{\ell=0,1,\dots} (-1)^{\ell-j} \binom{\ell}{j} \binom{k}{\ell} = \binom{k}{j} \frac{\sigma_k}{\sigma_j} \sum_{\ell=0}^{k-j} (-1)^\ell \binom{k-j}{\ell} = \delta_{j,k}. \quad (\text{A.15})$$

(3) *Three term recurrence formula.* From equations I-(3.55), (A.3) and (A.10), one has for $n = 0, 1, \dots$ ³¹,

$$H_{n+1} = (A_n x + B_n) H_n - C_n H_{n-1} \quad A_n = 2 \quad B_n = 0 \quad C_n = 2n. \quad (\text{A.16})$$

(4) *Differentiation formulae.* With w defined by equation (A.2) and f and g some polynomials, one has

$$w' = -2xw \quad (\text{A.17})$$

$$(f, g') = \int_{-\infty}^{\infty} (wf) dg = - \int_{-\infty}^{\infty} d(wf) g = -(f' - 2xf, g) \quad (\text{A.18})$$

$$(f'' - 2xf', g) = -(f', g') = (f, g'' - 2xg'). \quad (\text{A.19})$$

These equations can be used in two ways to generate differentiation formulae for $n = 0, 1, \dots$

• First method

(i) As any polynomial of degree $n-1$, $H'_n = \sum_{j=0}^{n-1} \alpha_j H_j$ where, from the orthogonality relation I-(A.2) and equation (A.18),

$$\alpha_j h_j = (H_j, H'_n) = -(H'_j - 2xH_j, H_n). \quad (\text{A.20})$$

²⁹ See, [3] table 22.12 where, the numerical values of $b_{m,n}$, and also $a_{m,n}$, are listed up to $n = 12$, but the general expressions (A.12)–(A.14) are not given there. Formulae (A.12) and (A.13) can be found in [11] chapter 6, exercise 6 p 339.

³⁰ See, e.g., [16] p 3, going from equation (1.8) to equation (1.91).

³¹ See, e.g., [4] 10.13 (5).

Since $H'_j - 2xH_j$ is a polynomial of degree $j + 1$, the scalar product above, and thus α_j , vanishes for $j = 0, \dots, n - 2$. Computing the coefficient of x^{n-1} in H'_n fixes $\alpha_{n-1} = nk_n/k_{n-1}$. One gets ³²,

$$H'_n = 2nH_{n-1}. \quad (\text{A.21})$$

Following the same steps, first order differentiation formula for $x^p H'_n$, with $p = 0, 1, \dots$, can thereby be obtained. Thus, for $p = 1$, considering the polynomial xH'_n of degree n , one finds

$$xH'_n = nH_n + 2n(n-1)H_{n-2}. \quad (\text{A.22})$$

(ii) Similarly, being a polynomial of degree n , $H''_n - 2xH'_n = \sum_{j=0}^n \alpha_j H_j$ where, from equation (A.19),

$$\alpha_j h_j = (H_j, H''_n - 2xH'_n) = (H''_n - 2xH'_n, H_n). \quad (\text{A.23})$$

Since $H''_j - 2xH'_j$ is a polynomial of degree j , α_j vanishes for $j = 0, \dots, n - 1$. Computing the coefficient of x^n in $H''_n - 2xH'_n$ fixes α_n . Thereby, H_n satisfies the *second order differential equation* (2.36) ³³,

$$A(x)H''_n + B(x)H'_n + \lambda_n H_n = 0 \quad A(x) = 1 \quad B(x) = -2x \quad \lambda_n = 2n. \quad (\text{A.24})$$

• Second method (permutating n and j in the relations above)

Only this method can be extended to the SBO polynomials as shown in section 2.3.

For $j = 0, \dots, n$, the polynomial H'_j of degree $j - 1$ is orthogonal to H_n , and with equation (A.18),

$$0 = (H_n, H'_j) = (H'_n - 2xH_n, H_j). \quad (\text{A.25})$$

Hence, the particular polynomial $H'_n - 2xH_n$ of degree $n + 1$ is orthogonal to the subspace \mathcal{P}_{n+1} , and thus it reads $\alpha_{n+1}H_{n+1}$. Computing the coefficient of x^{n+1} in $H'_n - 2xH_n$ fixes α_{n+1} . One gets

$$H'_n - 2xH_n = -H_{n+1}. \quad (\text{A.26})$$

Following the same steps, first order differentiation formula for $x^p H'_n$, with $p = 0, 1, \dots$, can thereby be obtained. Thus, with $p = 1$, considering the polynomial xH'_j of degree j , one finds

$$xH'_n = -\frac{1}{2}H_{n+2} + (2x^2 - 1 - n)H_n. \quad (\text{A.27})$$

Actually, all these differentiation formulae and the three term recurrence formula are not independent. Using simple algebra (substitutions and derivations), it can be checked that:

(i) all these differentiation formulae follow from the first order differentiation formula (A.21) and the three term recurrence formula (A.16);

(ii) more generally, among the equations (A.16), (A.21), (A.24) and (A.26), any pair of equations implies the other two equations. In particular, the first order differentiation formula, e.g., (A.26), and the second order differential equation (A.24) yield the recurrence equation (A.16). *This kind of method is used in section 2.4 to find a five term recurrence formula with respect to the degree for the Hermite SBO polynomials in the special case $w_2 := e^{-2x^2}$;*

(iii) note that the differentiation formulae (A.22) and (A.27) relate Hermite polynomials having the same parity. From both equations, one gets the recurrence formula,

$$H_{n+2} = 2(2x^2 - 2n - 1)H_n - 4n(n-1)H_{n-2} \quad (\text{A.28})$$

which can be obtained also iterating the three term recurrence formula (A.16).

³² See, e.g., [4] 10.13 (14).

³³ See, e.g., [4] 10.13 (12).

(5) *Generating function.* The most common generating function reads ³⁴,

$$F_z(x) := \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) z^n = e^{2xz-z^2}. \quad (A.29)$$

Appendix B. Laguerre polynomials

(1) *Definition.* (Generalized) Laguerre polynomials $L_n^{(\alpha)}$, $n = 0, 1, \dots$ are defined with the measure and normalization ³⁵,

$$a := 0 \quad b := \infty \quad w := e^{-x} x^\alpha \quad \alpha > -1 \quad (B.1)$$

$$k_n := \frac{(-1)^n}{n!} \quad n = 0, 1, \dots \quad (B.2)$$

(2) *Connection coefficients.* From equations (A.1) and the Euler integral (see footnote 27), the moment reads

$$c_j = \int_0^\infty dx e^{-x} x^{\alpha+j} = \Gamma(\alpha+1+j) \quad j = 0, 1, \dots \quad (B.3)$$

Then, all the determinants which occur in the general formulae I-(A.5)–I-(A.10) can be obtained with lemma D. For $n = 0, 1, \dots$ and $m = 0, \dots, n$, one finds ³⁶,

$$Z_n = \prod_{j=0}^n j! \Gamma(\alpha+1+j) \quad (B.4)$$

$$h_n = \frac{\Gamma(\alpha+1+n)}{n!} \quad (B.5)$$

$$a_{m,n} = \frac{(-1)^m}{n!} \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1+m)} \binom{n}{m} = \frac{(-1)^m}{m!} \binom{\alpha+n}{n-m} \quad (B.6)$$

$$\hat{r}_n = -n(\alpha+n) \quad \hat{s}_n = \frac{1}{2} n(n-1)(\alpha+n)(\alpha+n-1) \quad (B.7)$$

$$L_n^{(\alpha)}(0) = \frac{(\alpha+1)_n}{n!} = \binom{\alpha+n}{n}. \quad (B.8)$$

Similarly, the connection coefficient $b_{m,n}$, not given in the standard textbook already quoted, can be obtained from equations I-(A.10), (D.2) and (C.3). For $m = 0, \dots, n$ one has [3] ³⁷,

$$\begin{aligned} b_{m,n} &= \frac{k_m^{-1}}{m!} \sum_{\ell=0}^m (-1)^{m-\ell} \binom{m}{\ell} \frac{\Gamma(\alpha+1+n+\ell)}{\Gamma(\alpha+1+\ell)} \\ &= (-1)^m m! \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1+m)} \binom{n}{m} = (-1)^m n! \binom{\alpha+n}{n-m}. \end{aligned} \quad (B.9)$$

³⁴ See, e.g., [4] 10.13 (19).

³⁵ See, e.g., [4] 10.12 (1) and (2). According to [3] 22.2.12, α is written between parentheses to avoid any confusion with a power index.

³⁶ See, e.g., [4] 10.12 (2) for h_n , (7) for $a_{m,n}$, (2) for r_n only (see footnote 47 in paper I) and (13) for $L_n^{(\alpha)}(0)$.

³⁷ See, [3] table 22.10 where, the values of $b_{m,n}$, and also $a_{m,n}$, are listed only up to $n = 12$, but the general expression (B.9) is not given.

This value of $b_{m,n}$ can also be obtained directly with lemma A, setting $\rho_j = \Gamma(\alpha + 1 + j)/j!$ and $\sigma_j = \Gamma(\alpha + 1 + j)$.

(3) *Three term recurrence formula.* From equations I-(3.55), (B.2) and (B.7), one has for $n = 0, 1, \dots$ ³⁸,

$$\begin{aligned} L_{n+1}^{(\alpha)} &= (A_n x + B_n) L_n^{(\alpha)} - C_n L_{n-1}^{(\alpha)} \\ (n+1)A_n &= -1 \quad (n+1)B_n = \alpha + 2n + 1 \quad (n+1)C_n = \alpha + n. \end{aligned} \quad (B.10)$$

(4) *Differentiation formulae.* With w defined by equation (B.1) and f and g some polynomials, one has

$$(xw)' = -(x - \alpha - 1)w \quad (B.11)$$

$$(f, xg') = \int_0^\infty xwf dg = - \int_0^\infty d(xwf) g = -(xf' - (x - \alpha - 1)f, g) \quad (B.12)$$

$$(xf'' - (x - \alpha - 1)f', g) = -(f', xg') = -(xf', g') = (f, xg'' - (x - \alpha - 1)g'). \quad (B.13)$$

As for the Hermite polynomials in appendix A 4-, these relations can be used in two ways to generate differentiation formulae for $n = 0, 1, \dots$

- First method

(i) Considering the polynomial $xL_n^{(\alpha)'} with equation (B.12), one finds ³⁹,$

$$xL_n^{(\alpha)'} = nL_n^{(\alpha)} - (\alpha + n)L_{n-1}^{(\alpha)}. \quad (B.14)$$

(ii) Considering the polynomial $xL_n^{(\alpha)''} - (x - \alpha - 1)L_n^{(\alpha)'}$ with equation (B.13), one gets the *second order differential equation* (2.36) ⁴⁰,

$$A(x)L_n^{(\alpha)''} + B(x)L_n^{(\alpha)'} + \lambda_n L_n^{(\alpha)} = 0 \quad A(x) = x \quad B(x) = \alpha + 1 - x \quad \lambda_n = n. \quad (B.15)$$

- Second method (permutating n and j in the relations above). *Only this method can be extended to the SBO polynomials as shown in section 3.3.*

For $j = 0, \dots, n-1$, considering the polynomial $xL_j^{(\alpha)'}$, one finds ⁴¹,

$$xL_n^{(\alpha)'} = (n+1)L_{n+1}^{(\alpha)} + (x - \alpha - 1 - n)L_n^{(\alpha)}. \quad (B.16)$$

As for the Hermite polynomials, it can be checked that:

(i) all these differentiation formulae follow from the first order differentiation formula (B.14) and the three term recurrence formula (B.10);

(ii) more generally, among the equations (B.10) and (B.14)–(B.16), any pair of equations implies both other equations. In particular, the first order differentiation formula (B.16) and the second order differential equation (B.15) yield the recurrence equation (B.10). *This kind of method is used in section 3.4 to find a five term recurrence formula with respect to the degree for the Laguerre SBO polynomials in the special case $w_2 := e^{-2x}x^\alpha$.*

(5) *Generating function.* The most common generating function reads ⁴²,

$$F_z^{(\alpha)}(x) := \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n = (1-z)^{-\alpha-1} \exp \frac{xz}{z-1} \quad |z| < 1. \quad (B.17)$$

³⁸ See, e.g., [4] 10.12 (3).

³⁹ See, e.g., [4] 10.12 (12).

⁴⁰ See, e.g., [4] 10.12 (10).

⁴¹ See, e.g., [4] 10.12 (12).

⁴² See, e.g., [4] 10.12 (17).

(6) *Interrelations between Hermite and Laguerre polynomials.* For any polynomials f and g , a change of variable in the integral yields

$$\int_{-\infty}^{\infty} dx e^{-x^2} f(x)g(x) = \int_0^{\infty} dx e^{-x} x^{-\frac{1}{2}} f(\sqrt{x})g(\sqrt{x}) = \int_0^{\infty} dx e^{-x} x^{\frac{1}{2}} \frac{f(\sqrt{x})}{\sqrt{x}} \frac{g(\sqrt{x})}{\sqrt{x}}. \quad (B.18)$$

It follows that for $n = 0, 1, \dots$ ⁴³,

$$\widehat{L}_n^{(-\frac{1}{2})}(x) = \widehat{H}_{2n}(\sqrt{x}) \quad H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(-\frac{1}{2})}(x^2) \quad (B.19)$$

$$\widehat{L}_n^{(\frac{1}{2})}(x) = \frac{\widehat{H}_{2n}(\sqrt{x})}{\sqrt{x}} \quad H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{(\frac{1}{2})}(x^2). \quad (B.20)$$

Appendix C. Pochhammer symbol and some related relations

Definition and properties of the *Pochhammer symbol* or *rising factorial* $(z)_n$ are detailed in [13, 14]⁴⁴. With n a non-negative integer and z some complex number, one has⁴⁵,

$$(z)_n := \frac{\Gamma(z+n)}{\Gamma(z)} = (-1)^n n! \binom{-z}{n} = z(z+1) \cdots (z+n-1) = (-1)^n (-z-n+1)_n. \quad (C.1)$$

The Pochhammer symbol satisfies the *binomial formula*, i.e. with n a non-negative integer and z and w some complex numbers⁴⁶,

$$(z+w)_n = \sum_{\ell=0}^n \binom{n}{\ell} (z)_{\ell} (w)_{n-\ell}. \quad (C.2)$$

Lemma C.1. *With j and k non-negative integers and c some complex number not zero or a negative integer, one has*

$$\sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \frac{\Gamma(c+j+\ell)}{\Gamma(c+\ell)} = k! \binom{j}{k} \frac{\Gamma(c+j)}{\Gamma(c+k)} \quad (C.3)$$

where, the binomial coefficient $\binom{j}{k}$ vanishes if $j < k$.

Proof. Setting $n = k$, $z = c + j$ and $w = -c - k + 1$ in the binomial formula (C.2) yields

$$(j-k+1)_k = \sum_{\ell=0}^k \binom{k}{\ell} (c+j)_{\ell} (-c-k+1)_{k-\ell}. \quad (C.4)$$

Then, it follows from equation (C.1) that,

$$(j-k+1)_k = k! \binom{j}{k} \quad (c+j)_{\ell} = \frac{\Gamma(c+j+\ell)}{\Gamma(c+j)} \quad (-c-k+1)_{k-\ell} = (-1)^{k-\ell} \frac{\Gamma(c+k)}{\Gamma(c+\ell)} \quad (C.5)$$

thereby recovering equation (C.3) provided $c+j \neq 0, -1, \dots$. This identity can be proved also directly by recurrence on k .

⁴³ See, e.g., [4] 10.13 (2) and (3).

⁴⁴ See, e.g., [13] appendix I or [14] section 2.

⁴⁵ See, e.g., [3] 6.1.5, 6.1.21 and 6.1.22.

⁴⁶ See, e.g., [14] section 2.6.

Lemma C.2. With c , z and w some complex numbers, c not zero or a negative integer, and j and k non-negative integers, let,

$$S_1(j, k, c; z, w) := \sum_{p=0}^{\min(j, k)} \binom{j}{p} \binom{k}{p} \frac{p!}{\Gamma(c+p)} (zw)^p (1+z)^{j-p} (1+w)^{k-p} \quad (C.6)$$

$$S_2(j, k, c; z, w) := \sum_{p=0}^j \sum_{q=0}^k \binom{j}{p} \binom{k}{q} \frac{\Gamma(c+p+q)}{\Gamma(c+p)\Gamma(c+q)} z^p w^q \quad (C.7)$$

$$S(j, k, c; z, w) := \frac{(1+z)^j (1+w)^k}{\Gamma(c)} F\left(-j, -k; c; \frac{zw}{(1+z)(1+w)}\right) \quad (C.8)$$

where, F is an hypergeometric function. Then, one has

$$S(j, k, c; z, w) = S(k, j, c; w, z) = S_1(j, k, c; z, w) = S_2(j, k, c; z, w). \quad (C.9)$$

Proof. From equations (C.1), in terms of the Pochhammer symbol, one has

$$\begin{aligned} S_1(j, k, c; z, w) &= \frac{(1+z)^j (1+w)^k}{\Gamma(c)} \sum_{p=0}^{\min(j, k)} \frac{(-j)_p (-k)_p}{(c)_p} \frac{1}{p!} \left(\frac{zw}{(1+z)(1+w)} \right)^p \\ &= S(j, k, c; z, w) \end{aligned} \quad (C.10)$$

$$\begin{aligned} S_2(j, k, c; z, w) &= \frac{1}{\Gamma(c)} \sum_{p=0}^j \sum_{q=0}^k \frac{(-j)_p (-k)_q (c)_{p+q}}{(c)_p (c)_q} \frac{(-z)^p}{p!} \frac{(-w)^q}{q!} \\ &= \frac{1}{\Gamma(c)} F_2(c, -j, -k, c, c; -z, -w) \end{aligned} \quad (C.11)$$

where, F_2 is a special case of a generalized hypergeometric series of two variables listed by Horn and reducible to an hypergeometric function [12]⁴⁷,

$$F_2(c, -j, -k, c, c; -z, -w) = (1+z)^j (1+w)^k F\left(-j, -k; c; \frac{zw}{(1+z)(1+w)}\right). \quad (C.12)$$

The symmetry property $(j, z) \leftrightarrow (k, w)$ is obvious. This completes the proof of equation (C.9).

Corollary C. Special values of $S(j, k, c; z, w)$, see lemma C.2 equations (C.6)–(C.9):

$$S(j, k, c; z, -1) = \binom{j}{k} \frac{k!}{\Gamma(c+k)} (-z)^k (1+z)^{j-k} \quad (C.13)$$

$$S(j, k, c; -1, -1) = \frac{j!}{\Gamma(c+j)} \delta_{j,k} \quad (C.14)$$

$$S(j, k, c; z, -z-1) = (1+z)^j (-z)^k \frac{\Gamma(c+j+k)}{\Gamma(c+j)\Gamma(c+k)} \quad (C.15)$$

where, in equation (C.13), the binomial coefficient $\binom{j}{k}$ vanishes if $j < k$.

Proof. For $z = -1$ or $w = -1$, the first two equations follow directly from lemma C.2. Note that S_1 and S_2 have no singularity, in particular at $z = -1$ or/and $w = -1$. If $w = -1$, only the term $p = k$ contributes

⁴⁷ See, e.g., [12] 5.7.1 (7) and 5.10 (3).

to equation (C.6), giving equation (C.13). Alternately, setting $w = -1$ in equation (C.7), the sum over q can be performed with lemma C.1. For $z = -w - 1$, one has $zw/((1+z)(1+w)) = 1$, then from the *Gauss summation theorem* ⁴⁸,

$$F(-j, -k, c; 1) = \frac{\Gamma(c)\Gamma(c+j+k)}{\Gamma(c+j)\Gamma(c+k)} \quad (C.16)$$

ending the proof of equation (C.15).

Appendix D. Some determinants with gamma functions as entries

Lemma D. *With n a positive integer and c some complex number not zero or a negative integer, let \mathbf{M} be an $n \times n$ matrix which element $M_{j,k}$ is equal to $\Gamma(c+j+k)$, except possibly for the last row,*

$$\mathbf{M} := \begin{pmatrix} (\Gamma(c+j+k))_{\substack{j=0,\dots,n-2 \\ k=0,\dots,n-1}} \\ (M_{n-1,k})_{k=0,\dots,n-1} \end{pmatrix}. \quad (D.1)$$

Then, one has

$$\frac{\det \mathbf{M}}{\prod_{j=0}^{n-2} j! \Gamma(c+j)} = \Gamma(c+n-1) \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} \binom{n-1}{\ell} \frac{1}{\Gamma(c+\ell)} M_{n-1,\ell} \quad (D.2)$$

$$\det [\Gamma(c+j+k)]_{j,k=0,\dots,n-1} = \prod_{j=0}^{n-1} j! \Gamma(c+j). \quad (D.3)$$

Proof. A determinant is not changed if one adds to any column a linear combination of other columns. Replacing for $k = 0, \dots, n-1$ the column \mathcal{C}_k of $\det \mathbf{M}$ by,

$$\mathcal{C}_k + \sum_{\ell=0}^{k-1} (-1)^{k-\ell} \binom{k}{\ell} \frac{\Gamma(c+k)}{\Gamma(c+\ell)} \mathcal{C}_\ell = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \frac{\Gamma(c+k)}{\Gamma(c+\ell)} \mathcal{C}_\ell \quad (D.4)$$

yields a new matrix \mathbf{M}' with an equal determinant and which is triangular. Indeed, from definition of \mathbf{M} and lemma C.1, one has for $j = 0, \dots, n-2$ and $k = 0, \dots, n-1$,

$$M'_{j,k} = \Gamma(c+k) \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \frac{\Gamma(c+j+\ell)}{\Gamma(c+\ell)} = k! \binom{j}{k} \Gamma(c+j) \quad (D.5)$$

which vanishes for $j < k$. Then, the determinant is the product of the diagonal elements $j! \Gamma(c+j)$ for $j = 0, \dots, n-2$ and $M'_{n-1,n-1}$. From equation (D.4), this last element reads

$$M'_{n-1,n-1} = \Gamma(c+n-1) \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} \binom{n-1}{\ell} \frac{1}{\Gamma(c+\ell)} M_{n-1,\ell} \quad (D.6)$$

completing the proof of equation (D.2). If moreover, $M_{n-1,k} = \Gamma(c+n-1+k)$, using again the equation (D.5) for $j = k = n-1$ yields the result (D.3) already published in [15, 14] ⁴⁹.

Appendix E. Special cases of Hermite standard bloc orthogonal polynomials

With the weight functions, $x \in (-\infty, \infty)$, $w := e^{-x^2}$ and $w_2 := e^{-2x^2}$, one has:

⁴⁸ See, e.g., [3] 15.1.20 or [13] (1.7.6).

⁴⁹ See [15] equation (A.12) or [14] equation (4.5).

$$i = 0 \quad n = 1, 2, \dots$$

$$\hat{P}_{0;1} = x \quad (E.1)$$

$$\hat{P}_{0;2} = x^2 - \frac{1}{4} \quad (E.2)$$

$$\hat{P}_{0;3} = x^3 - \frac{3}{4}x \quad (E.3)$$

$$\hat{P}_{0;4} = x^4 - \frac{3}{2}x^2 + \frac{3}{16} \quad (E.4)$$

$$\hat{P}_{0;5} = x^5 - \frac{5}{2}x^3 + \frac{15}{16}x; \quad (E.5)$$

$$i = 1 \quad n = 1, 2, \dots^{50},$$

$$\hat{P}_{1;2} = x^2 - \frac{1}{2} \quad (E.6)$$

$$\hat{P}_{1;4} = x^4 - \frac{7}{4}x^2 + \frac{1}{8} \quad (E.7)$$

$$\hat{P}_{1;6} = x^6 - 4x^4 + \frac{47}{16}x^2 - \frac{11}{32}; \quad (E.8)$$

$$i = 2 \quad n = 2, 3, \dots^{51},$$

$$\hat{P}_{2;3} = x^3 - \frac{3}{2}x \quad (E.9)$$

$$\hat{P}_{2;5} = x^5 - \frac{13}{4}x^3 + \frac{9}{8}x. \quad (E.10)$$

If, instead of w and w_2 , one considers the following weight functions,

$$w^{(c)}(x) := w(\sqrt{c}x) = e^{-cx^2} \quad w_2^{(c)} := w_2(\sqrt{c}x) = e^{-2cx^2} \quad c > 0 \quad (E.11)$$

then, changing x into $\sqrt{c}x$ in the integral defining the scalar product $(\cdot, \cdot)_2$, the corresponding monic Hermite SBO polynomials $\hat{P}_{i;n}^{(c)}$ and the orthogonality constant $\hat{H}_{i;n}^{(c)}$ read

$$\hat{P}_{i;n}^{(c)}(x) = c^{-\frac{1}{2}n} \hat{P}_{i;n}(\sqrt{c}x) \quad \hat{H}_{i;n}^{(c)} = c^{-(\frac{1}{2}+n)} \hat{H}_{i;n}. \quad (E.12)$$

⁵⁰ In the case $i = 1$, ‘Hermite polynomials under a constraint of zero average’ have already been considered in [6] and [8], with different choices of weight functions and normalization:

- in [6] (1) and appendix A, the polynomials $A_n := 2^n x^n + O(x^{n-1})$, $n = 1, 2, \dots$, with $w := e^{-\frac{1}{2}x^2}$ and $w_2 := e^{-x^2}$. From equation (E.12) below with $c = \frac{1}{2}$, one gets $A_n(x) = 2^{\frac{3}{2}n} \hat{P}_{1;n}(x/\sqrt{2})$, $n = 1, 2, \dots$;

- in [8] (43), the polynomials $Q_{2n} := 2^{n-1} x^n + O(x^{n-1})$, $n = 1, 2, \dots$, with $w := e^{-\frac{1}{2}x^2}$ and $w_2 := e^{-x^2}$. Then, one has $Q_{2n} = 2^{-(n+1)} A_{2n}$, $n = 1, 2, \dots$. Note that with this choice of w_2 , one has $\hat{Q}_{2;n} = \hat{H}_n$, $n = 0, 1, \dots$;

- in [6] section 4, the polynomials $G_n := c_n \hat{P}_{1;n}$, $n = 1, 2, \dots$, with $(G_m, G_n)_2 = \sqrt{\pi/2} \delta_{m,n}$, $w := e^{-x^2}$ and $w_2 := e^{-2x^2}$. From equations (2.18) and (2.19), one finds $c_{2n+1} = \pm 2^{2n+1}/\sqrt{(2n+1)!}$, $n = 0, 1, \dots$ and $c_{2n} = \pm 2^{2n+1} \sqrt{2n}/\sqrt{(2n+1)!}$, $n = 1, 2, \dots$;

- in [8] (44), (45a)–(45d), the polynomials $\Gamma_n := |c_n| \hat{P}_{1;n}$, $n = 1, 2, \dots$, with c_n as for G_n above, $(\Gamma_m, \Gamma_n)_2 = \sqrt{\pi/2} \delta_{m,n}$, $w := e^{-x^2}$ and $w_2 := e^{-2x^2}$, and in addition, with a positive coefficient of x^n in Γ_n .

⁵¹ The ‘Hermite polynomials constrained by a zero momentum’, considered in [6] section 3 (17), with $w := e^{-\frac{1}{2}x^2}$ and $w_2 := e^{-x^2}$, is not the case $i = 2$. Indeed, the one-dimensional subspace \mathcal{E}_1 considered there, and spanned by $\{x^1\}$, is not \mathcal{P}_2 spanned by $\{x^0, x^1\}$. Nevertheless, as discussed in the conclusion I-4 (see footnote 44 in paper I), from parity arguments, it follows that in this non-standard case, the block orthogonal polynomials \hat{P}_n , $n = 0, 2, 3, \dots$ are given by, $\hat{P}_{2n} = \hat{Q}_{2;2n}$, $n = 0, 1, \dots$ and $\hat{P}_{2n+1} = \hat{P}_{2;2n+1}$, $n = 0, 1, \dots$. From equation (E.12) with $c = \frac{1}{2}$, one gets with $B_n := 2^n x^n + O(x^{n-1})$, $n = 0, 2, 3, \dots$, $B_{2n} = H_{2n}$, $n = 0, 1, \dots$ and $B_{2n+1}(x) = 2^{3(n+\frac{1}{2})} \hat{P}_{2;2n+1}(x/\sqrt{2})$, $n = 1, 2, \dots$

Appendix F. Special cases of Laguerre standard bloc orthogonal polynomials

With the weight functions, $x \in [0, \infty)$, $w := e^{-x}x^\alpha$ and $w_2 := e^{-2x}x^\alpha$, $\alpha > -1$, one has:

$i = 0 \quad n = 1, 2, \dots$

$$\hat{P}_{0;1} = x - \frac{1}{2}(\alpha + 1) \quad (F.1)$$

$$\hat{P}_{0;2} = x^2 - (\alpha + 2)x + \frac{1}{4}(\alpha + 1)(\alpha + 2) \quad (F.2)$$

$$\hat{P}_{0;3} = x^3 - \frac{3}{2}(\alpha + 3)x^2 + \frac{3}{4}(\alpha + 2)(\alpha + 3)x - \frac{1}{8}(\alpha + 1)(\alpha + 2)(\alpha + 3) \quad (F.3)$$

$$\begin{aligned} \hat{P}_{0;4} = & x^4 - 2(\alpha + 4)x^3 + \frac{3}{2}(\alpha + 3)(\alpha + 4)x^2 - \frac{1}{2}(\alpha + 2)(\alpha + 3)(\alpha + 4)x \\ & + \frac{1}{16}(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4); \end{aligned} \quad (F.4)$$

$i = 1 \quad n = 1, 2, \dots$ ⁵²,

$$\hat{P}_{1;1} = x - (\alpha + 1) \quad (F.5)$$

$$\hat{P}_{1;2} = x^2 - \frac{1}{2}(3\alpha + 5)x + \frac{1}{2}(\alpha + 1)^2 \quad (F.6)$$

$$\hat{P}_{1;3} = x^3 - (2\alpha + 5)x^2 + \frac{1}{4}(5\alpha^2 + 19\alpha + 20)x - \frac{1}{4}(\alpha + 1)(\alpha^2 + 3\alpha + 4) \quad (F.7)$$

$$\begin{aligned} \hat{P}_{1;4} = & x^4 - \frac{1}{2}(5\alpha + 17)x^3 + \frac{3}{4}(3\alpha^2 + 17\alpha + 26)x^2 \\ & - \frac{1}{8}(7\alpha^3 + 48\alpha^2 + 125\alpha + 108)x + \frac{1}{8}(\alpha + 1)^2(\alpha^2 + 5\alpha + 12) \end{aligned} \quad (F.8)$$

$$\begin{aligned} \hat{P}_{1;5} = & x^5 - (3\alpha + 13)x^4 + \frac{1}{2}(7\alpha^2 + 53\alpha + 106)x^3 - \frac{1}{2}(4\alpha^3 + 39\alpha^2 + 137\alpha + 162)x^2 \\ & + \frac{1}{16}(9\alpha^4 + 98\alpha^3 + 447\alpha^2 + 886\alpha + 648)x \\ & - \frac{1}{16}(\alpha + 1)(\alpha^4 + 10\alpha^3 + 47\alpha^2 + 86\alpha + 72); \end{aligned} \quad (F.9)$$

$i = 2 \quad n = 2, 3, \dots$

$$\hat{P}_{2;2} = x^2 - 2(\alpha + 2)x + (\alpha + 1)(\alpha + 2) \quad (F.10)$$

$$\hat{P}_{2;3} = x^3 - \frac{1}{2}(5\alpha + 13)x^2 + 2(\alpha + 2)^2x - \frac{1}{2}(\alpha + 1)^2(\alpha + 2) \quad (F.11)$$

$$\begin{aligned} \hat{P}_{2;4} = & x^4 - (3\alpha + 10)x^3 + \frac{1}{4}(13\alpha^2 + 71\alpha + 102)x^2 \\ & - \frac{1}{2}(\alpha + 2)(3\alpha^2 + 13\alpha + 18)x + \frac{1}{4}(\alpha + 1)(\alpha + 2)(\alpha^2 + 3\alpha + 6) \end{aligned} \quad (F.12)$$

$$\begin{aligned} \hat{P}_{2;5} = & x^5 - \frac{1}{2}(7\alpha + 29)x^4 + \frac{1}{4}(19\alpha^2 + 135\alpha + 254)x^3 \\ & - \frac{1}{8}(25\alpha^3 + 222\alpha^2 + 719\alpha + 810)x^2 + \frac{1}{2}(\alpha + 2)^2(2\alpha^2 + 11\alpha + 27)x \end{aligned}$$

⁵² In the case $i = 1$, ‘modification of Laguerre polynomials by a constraint of zero average’ have already been considered in [7] section 2, with different choices of weight functions and normalization: the monic polynomials G_n^d , $n = 1, 2, \dots$, with $\alpha := d - 1$, $w := e^{-\frac{1}{2}x}x^{d-1}$ and $w_2 := e^{-x}x^{d-1}$. Setting $c = \frac{1}{2}$ in equation (F.16) below yields: $G_n^d(x) = 2^n \hat{P}_{1;n}(x/2)|_{\alpha=d-1}$, $n = 1, 2, \dots$. Note that with this choice of w_2 , $\hat{Q}_{2;n} = \hat{L}_n^{(\alpha)}$, $n = 0, 1, \dots$

$$-\frac{1}{8}(\alpha+1)^2(\alpha+2)(\alpha^2+5\alpha+18) \quad (F.13)$$

$$\begin{aligned} \widehat{P}_{2,6} = & x^6 - 4(\alpha+5)x^5 + \frac{1}{2}(13\alpha^2+115\alpha+268)x^4 - \frac{1}{2}(11\alpha^3+127\alpha^2+526\alpha+752)x^3 \\ & + \frac{1}{16}(41\alpha^4+538\alpha^3+2947\alpha^2+7418\alpha+7056)x^2 \\ & - \frac{1}{8}(\alpha+2)(5\alpha^4+58\alpha^3+319\alpha^2+770\alpha+720)x \\ & + \frac{1}{16}(\alpha+1)(\alpha+2)(\alpha^4+10\alpha^3+59\alpha^2+122\alpha+144). \end{aligned} \quad (F.14)$$

If, instead of w and w_2 , one considers the following weight functions,

$$w^{(c)}(x) := c^{-\alpha}w(cx) = e^{-cx}x^\alpha \quad w_2^{(c)}(x) := c^{-\alpha}w_2(cx) = e^{-2cx}x^\alpha \quad c > 0 \quad (F.15)$$

then, changing x into cx in the integral defining the scalar product $(\cdot, \cdot)_2$, the corresponding monic Laguerre SBO polynomials $\widehat{P}_{i;n}^{(c)}$ and the orthogonality constant $\widehat{H}_{i;n}^{(c)}$ read

$$\widehat{P}_{i;n}^{(c)}(x) = c^{-n}\widehat{P}_{i;n}(cx) \quad \widehat{H}_{i;n}^{(c)} = c^{-(\alpha+1+2n)}\widehat{H}_{i;n}. \quad (F.16)$$

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